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Euclidean Gibbs Measures of Interacting Quantum Anharmonic Oscillators

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A rigorous description of the equilibrium thermodynamic properties of an infinite system of interacting ν-dimensional quantum anharmonic oscillators is given. The oscillators are indexed by the elements of a countable set $\mathbb{L} \subset \mathbb{R}^d$, possibly irregular; the anharmonic potentials vary from site to site and the interaction has infinite range. The description is based on the representation of the Gibbs states in terms of path measures—the so called Euclidean Gibbs measures. It is proven that: (a) the set of such measures $\mathcal{G}^{\rm t}$ is non-void and compact; (b) every $\mu \in \mathcal{G}^{\rm t}$ obeys an exponential integrability estimate, the same for the whole set $\mathcal{G}^{\rm t}$; (c) every $\mu \in \mathcal{G}^{\rm t}$ has a Lebowitz-Presutti type support; (d) $\mathcal{G}^{\rm t}$ is a singleton at high temperatures. The case of attractive interaction and $\nu = 1$ is studied in more detail. We prove that: (a) $|\mathcal{G}^{\rm t}| > 1$ at low temperatures; (b) $|\mathcal{G}^{\rm t}| = 1$ due to quantum effects and at a nonzero external field. Thereby, a qualitative theory of phase transitions and quantum effects, which interprets most important experimental data known for the corresponding physical objects, is developed.

KEY WORDS: Dobrushin-Lanford-Ruelle approach, Gibbs state, KMS state, temperature loop spaces, Dobrushin criteria, Feynman-Kac formula, quantum anharmonic crystal, Lee-Yang theorem, quantum effects, phase transitions

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1. INTRODUCTION

The quantum anharmonic oscillator is a mathematical model of a localized quantum particle moving in a potential field with sufficient growth at infinity and possibly multiple minima. Infinite systems of interacting quantum anharmonic oscillators possess interesting properties connected with the possibility of ordering caused by the interaction as well as with quantum stabilization competing the ordering. Most of the systems of this kind are related with solids, such as ionic crystals containing localized light particles oscillating in the field created by heavy ionic complexes, or quantum crystals consisting entirely of such particles. For instance, a potential field with multiple minima is seen by a helium atom located at the center of the crystal cell in bcc helium.[46] The same situation exists in other quantum crystals, He, H_2 and to some extent Ne. An example of the ionic crystal with localized quantum particles moving in a double-well potential field is a KDP-type ferroelectric with hydrogen bounds, in which such particles are protons or deuterons performing one-dimensional oscillations along the bounds. In this case the particle carries electric charge and its displacement produces dipole moment that should be reflected in the choice of the interparticle interaction. It is believed that structural phase transitions in such ferroelectrics are triggered by the ordering of protons.[21,83–85] Another relevant physical object of this kind is a system of light atoms, like Li, doped into ionic crystals, like KCl. The particles in this system are not necessarily regularly distributed. At last, quantum anharmonic oscillators are used as parts of the models describing interaction of vibrating quantum particles with a radiation (photon) field[40,68] or strong electron-electron correlations caused by the interaction of electrons with vibrating ions responsible for such phenomena as superconductivity, charge density waves, etc, see Refs. 32, 33. Thus, infinite systems of interacting quantum anharmonic oscillators are quite important models and their rigorous description is still a challenging mathematical task.

The model we consider has the following heuristic Hamiltonian

$$H = -\frac{1}{2}\sum_{\ell,\ell'} J_{\ell\ell'} \cdot (q_\ell, q_{\ell'}) + \sum_{\ell} H_\ell, \tag{1.1}$$

in which the interaction term is of dipole-dipole type. The sums run through a countable set $\mathbb{L} \subset \mathbb{R}^d$, the displacement q_ℓ is a ν-dimensional vector. The interaction intensities are supposed to be such that

$$J_{\ell\ell} = 0, \quad J_{\ell\ell'} = J_{\ell'\ell} \in \mathbb{R}, \quad \ell, \ell' \in \mathbb{L}. \tag{1.2}$$

By $(\cdot, \cdot)$ and $|\cdot|$ we denote the Euclidean scalar product and norm in $\mathbb{R}^\nu$. The Hamiltonian

$$H_\ell = H_\ell^{\rm har} + V_\ell(q_\ell) \stackrel{\rm def}{=} \frac{1}{2m}|p_\ell|^2 + \frac{a}{2}|q_\ell|^2 + V_\ell(q_\ell), \quad a > 0, \tag{1.3}$$

describes an isolated anharmonic oscillator of mass m and momentum p_ℓ. Its part $H_\ell^{\rm har}$ corresponds to a ν-dimensional quantum harmonic oscillator of rigidity a. The anharmonic potentials V_ℓ, which may vary from site to site, are supposed to obey certain uniform bounds responsible for the stability of the whole system. We do not assume that the interaction possesses special properties like translation invariance or has finite range. Therefore, our model describes also systems with long-range interactions and with spacial irregularities, e.g., caused by impurities or random components.

A complete description of the equilibrium thermodynamic properties of infinite-particle systems may be made by constructing their Gibbs states. Usually, Gibbs states of quantum models are defined as positive normalized functionals on algebras of observables, satisfying the Kubo-Martin-Schwinger (KMS) condition, see Ref. 23, which reflects the consistency between the dynamic and thermodynamic properties of the system proper to the thermodynamic equilibrium. For a subsystem located in a finite $\Lambda \subset \mathbb{L}$ and thus described by the local Hamiltonian H_Λ, the KMS condition is formulated by means of the unitary operators $\exp(\imath t H_\Lambda)$, $t \in \mathbb{R}$. To describe the dynamics of the whole model one has to take the infinite volume limits of $\exp(\imath t H_\Lambda)$, which certainly exist for finite rank H_Λ, e.g., for spin models. However for our model, such limits do not exist and therefore the KMS condition for the whole system cannot be formulated. This produces a fundamental problem and actually there is no canonical way to define Gibbs states, and hence to give a complete description of the thermodynamic properties of models like (1.1). The aim of this work is to bridge this gap with the help of path integrals.

In Ref. 1, an approach employing the fact that the local Hamiltonians H_Λ generate stochastic processes has been initiated. In this approach, the description of the local Gibbs states, based on the properties of the semi-group $\exp(-\tau H_\Lambda)$, $\tau > 0$, is translated into a "probabilistic language," that opens the possibility to apply here corresponding concepts and techniques. In this language, our model is the system of infinite dimensional "spins" ω_ℓ, $\ell \in \mathbb{L}$, being continuous paths $\omega_\ell : [0, \beta] \to \mathbb{R}^\nu$, $\omega_\ell(0) = \omega_\ell(\beta)$, called also temperature loops. Each spin is described by the path measure of the β-periodic Ornstein-Uhlenbeck process corresponding to H_ℓ^{har} multiplied by a density obtained from the anharmonic potential with the help of the Feynman-Kac formula. Afterwards, finite subsystems are associated with conditional probability measures, which by the Dobrushin-Lanford-Ruelle (DLR) equation determine the set of Gibbs measures $\mathcal{G}^{\mathrm{t}}$. This approach is called Euclidean due to its conceptual analogy with the Euclidean quantum field theory. Its further development was conducted in the papers.[2–8,11–14,16,48–50,52,54,55,66,67] Among the achievements one has to mention the settlement in Refs. 3, 5, 6 of a long standing problem of the influence of quantum effects on structural phase transitions in quantum anharmonic crystals, which first was discussed in Ref. 77, see also Refs. 67, 86, 87.

In the present article, we give a complete description of the set $\mathcal{G}^{\mathrm{t}}$ for the model (1.1) and hence essentially finalize the development of the Euclidean approach for such models. Our results fall into two groups of theorems. The first group describes the general case where $J_{\ell\ell'}$ and V_ℓ satisfy natural stability conditions only. We state that: $\mathcal{G}^{\mathrm{t}}$ is non-void and compact (Theorem 3.1); the elements of $\mathcal{G}^{\mathrm{t}}$ obey certain exponential moment estimates (Theorem 3.2) and have a Lebowitz-Presutti type support (Theorem 3.3); $\mathcal{G}^{\mathrm{t}}$ is a singleton at high temperatures (Theorem 3.4). The second group of theorems describes the case of $\nu = 1$ and $J_{\ell\ell'} \geq 0$. Here we employ the FKG order and show that the set $\mathcal{G}^{\mathrm{t}}$ has maximal and minimal elements (Theorem 3.8). If the model is translation invariant, we prove that the limiting pressure exists and is the same in all states (Theorem 3.10). Then under natural additional conditions on V_ℓ we show (Theorem 3.12) that the model undergoes a phase transition (for $d \geq 3$) and, on the other hand, $\mathcal{G}^{\rm t}$ is a singleton at all temperatures if a quantum stabilization condition is satisfied (Theorem 3.13). Finally, we describe a class of anharmonic potentials V_ℓ, for which $\mathcal{G}^{\rm t}$ is a singleton at a non-zero external field (Theorem 3.14). Here we use a version of the Lee-Yang theorem,[52] adapted to path measures. All these results are novel both for the quantum model and its classical analogs.

The paper is organized as follows. In Sec. 2 we describe the model in detail (Subsec. 2.1) and present the basic elements of the Euclidean approach (Subsec. 2.2 and 2.3). Afterwards, we introduce tempered configurations, a local Gibbs specification, and tempered Euclidean Gibbs measures of our model. In Sec. 3 we give the results in the form of the theorems described above. Comments, which in particular relate these results with those known in the literature, conclude the section. The remaining part of the article is dedicated to the proof of the theorems and is quite technical. We begin it by studying in detail the properties of the local Gibbs specification.

2. EUCLIDEAN GIBBS MEASURES

2.1. The Model

The infinite system of quantum oscillators we consider is described by the formal Hamiltonian (1.1), (1.3), defined on the set $\mathbb{L} \subset \mathbb{R}^d$, $d \in \mathbb{N}$. This set is equipped with the Euclidean distance $|\ell - \ell'|$ inherited from $\mathbb{R}^d$. We suppose that

$$\sup_{\ell \in \mathbb{L}} \sum_{\ell' \in \mathbb{L}} \frac{1}{(1 + |\ell - \ell'|)^{d+\epsilon}} < \infty, \tag{2.1}$$

for every $\epsilon > 0$. This is a kind of uniform regularity, which in particular means that big amounts of the elements of $\mathbb{L}$ cannot concentrate in the subsets of $\mathbb{R}^d$ of small volume. If $\mathbb{L}$ is a crystalline lattice the model is called *the quantum anharmonic crystal.* For simplicity, we shall always assume that $\mathbb{L} = \mathbb{Z}^d$ if $\mathbb{L}$ is a lattice.

Subsets of $\mathbb{L}$ are denoted by Λ. As usual, $|\Lambda|$ stands for the cardinality of Λ and Λ^c – for its complement $\mathbb{L} \setminus \Lambda$. We write $\Lambda \Subset \mathbb{L}$ if Λ is non-void and finite. By $\mathcal{L}$ we denote a cofinal (ordered by inclusion and exhausting the lattice) sequence of finite subsets of $\mathbb{L}$. Limits taken along such $\mathcal{L}$ are denoted by $\lim_{\mathcal{L}}$. We write $\lim_{\Lambda \nearrow \mathbb{L}}$ if the limit is taken along an unspecified sequence of this type. If we say that something holds for all ℓ, we mean that it holds for all $\ell \in \mathbb{L}$; expressions like $\sum_\ell$ mean $\sum_{\ell \in \mathbb{L}}$. By $(\cdot, \cdot)$ and $|\cdot|$, we denote the Euclidean scalar product and norm in all spaces like $\mathbb{R}^\nu$, $\mathbb{R}^d$; $\mathbb{N}_0$ will stand for the set of nonnegative integers.

The Hamiltonian (1.1) has no direct mathematical meaning and is "represented" by the local Hamiltonians H_Λ, $\Lambda \Subset \mathbb{L}$, which are

$$\begin{aligned} H_\Lambda &= \sum_{\ell \in \Lambda} \left[H_\ell^{\mathrm{har}} + V_\ell(q_\ell) \right] - \frac{1}{2} \sum_{\ell, \ell' \in \Lambda} J_{\ell\ell'}(q_l, q_{\ell'}) \\ &= \frac{1}{2m} \sum_{\ell \in \Lambda} |p_\ell|^2 + W_\Lambda(q_\Lambda), \quad q_\Lambda = (q_\ell)_{\ell \in \Lambda}. \end{aligned} \tag{2.2}$$

In the latter formula the first term is the kinetic energy; the potential energy is

$$W_\Lambda(q_\Lambda) = -\frac{1}{2} \sum_{\ell, \ell' \in \Lambda} J_{\ell\ell'}(q_\ell, q_{\ell'}) + \sum_{\ell \in \Lambda} \left[(a/2)|q_\ell|^2 + V_\ell(q_\ell) \right]. \tag{2.3}$$

The anharmonic potentials V_ℓ and the interaction intensities $J_{\ell\ell'}$ are subject to the following

Assumption 2.1. *All $V_\ell : \mathbb{R}^\nu \to \mathbb{R}$ are continuous and such that $V_\ell(0) = 0$; there exist $r > 1$, $A_V > 0$, $B_V \in \mathbb{R}$, and a continuous function $V : \mathbb{R}^\nu \to \mathbb{R}$, $V(0) = 0$, such that for all ℓ and $x \in \mathbb{R}^\nu$,*

$$A_V |x|^{2r} + B_V \leq V_\ell(x) \leq V(x). \tag{2.4}$$

We also assume that

$$\hat{J}_0 \stackrel{\mathrm{def}}{=} \sup_\ell \sum_{\ell'} |J_{\ell\ell'}| < \infty. \tag{2.5}$$

The lower bound in (2.4) is responsible for confining each particle in the vicinity of its equilibrium position. The upper bound is to guarantee that the oscillations of the particles located far from the origin are not suppressed. An example of V_ℓ to bear in mind is the polynomial

$$V_\ell(x) = \sum_{s=1}^{r} b_\ell^{(s)} |x|^{2s} - (h, x), \quad b_\ell^{(s)} \in \mathbb{R}, \ \ b_\ell^{(r)} > 0, \ \ r \geq 2, \tag{2.6}$$

in which $h \in \mathbb{R}^\nu$ is an external field and the coefficients $b_\ell^{(s)}$ vary in certain intervals, such that both estimates (2.4) hold. Under Assumption 2.1 H_Λ is a self-adjoint lower bounded operator in $L^2(\mathbb{R}^{\nu|\Lambda|})$ having discrete spectrum. It generates a positivity preserving semigroup such that

$$\mathrm{trace}[\exp(-\tau H_\Lambda)] < \infty, \quad \text{for all } \tau > 0. \tag{2.7}$$

Definition 2.2. The model is ferroelectric[3] if $J_{\ell\ell'} \geq 0$ for all ℓ, ℓ'. The interaction has finite range if there exists $R > 0$ such that $J_{\ell\ell'} = 0$ whenever $|\ell - \ell'| > R$. The model is translation invariant if $\mathbb{L}$ is a lattice, $V_\ell = V$ for all ℓ, and the matrix $(J_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ is invariant under translations of $\mathbb{L}$. The model is rotation invariant if for every orthogonal transformation $U \in O(\nu)$ and every ℓ, $V_\ell(Ux) = V_\ell(x)$.

If $V_\ell \equiv 0$ for all ℓ, one gets a system of interacting quantum harmonic oscillators, a quantum harmonic crystal if $\mathbb{L}$ is a lattice. It is stable if $\hat{J}_0 < a$, see Remark 3.5. below.

2.2. Quantum Gibbs States in the Euclidean Approach

Here we outline the basic elements of the Euclidean approach we apply in this article. More details can be found in Refs. 4, 7.

For $\Lambda \Subset \mathbb{L}$, the Hamiltonian H_Λ, defined by (2.2), acts in the physical Hilbert space $\mathcal{H}_\Lambda \stackrel{\mathrm{def}}{=} L^2(\mathbb{R}^{\nu|\Lambda|})$. In view of (2.7), one can introduce the local Gibbs state

$$\mathfrak{C}_\Lambda \ni A \mapsto \varrho_\Lambda(A) \stackrel{\mathrm{def}}{=} \frac{\mathrm{trace}(Ae^{-\beta H_\Lambda})}{\mathrm{trace}(e^{-\beta H_\Lambda})}, \tag{2.8}$$

which is a positive normalized functional on the algebra $\mathfrak{C}_\Lambda$ of all bounded linear operators (observables) on $\mathcal{H}_\Lambda$. The mappings

$$\mathfrak{C}_\Lambda \ni A \mapsto \mathfrak{a}_t^\Lambda(A) \stackrel{\mathrm{def}}{=} e^{itH_\Lambda} A e^{-itH_\Lambda}, \quad t \in \mathbb{R}, \tag{2.9}$$

constitute the group of time automorphisms which describes the dynamics of the system in Λ. The state ϱ_Λ satisfies the KMS (thermal equilibrium) condition relative to the dynamics $\mathfrak{a}_t^\Lambda$, see Definition 1.1 in Ref. 44. Multiplication operators by bounded continuous functions act as

$$(F\psi)(x) = F(x) \cdot \psi(x), \quad \psi \in \mathcal{H}_\Lambda, \quad F \in C_{\mathrm{b}}(\mathbb{R}^{\nu|\Lambda|}).$$

One can prove, see Ref. 55, that the linear span of the products

$$\mathfrak{a}_{t_1}^\Lambda(F_1) \cdots \mathfrak{a}_{t_n}^\Lambda(F_n), \tag{2.10}$$

with all possible choices of $n \in \mathbb{N}$, $t_1, \ldots, t_n \in \mathbb{R}$ and $F_1, \ldots, F_n \in C_{\mathrm{b}}(\mathbb{R}^{\nu|\Lambda|})$, is σ-weakly dense in $\mathfrak{C}_\Lambda$. Therefore, as a σ-weakly continuous functional (see page 65 of the first volume of Ref. 23), the state (2.8) is fully determined by its values on (2.10), that is, by *the Green functions*

$$G^\Lambda_{F_1,\ldots,F_n}(t_1, \ldots, t_n) \stackrel{\mathrm{def}}{=} \varrho_\Lambda\left[\mathfrak{a}_{t_1}^\Lambda(F_1) \cdots \mathfrak{a}_{t_n}^\Lambda(F_n)\right]. \tag{2.11}$$

[3] Usually such a model is called ferromagnetic; we adopt the above terminology in view of the ferroelectric interpretation mentioned in Introduction.

They can be considered as restrictions of functions $G^{\Lambda}_{F_1,\ldots,F_n}(z_1,\ldots,z_n)$, analytic in the domain

$$\mathcal{D}^n_\beta = \{(z_1,\ldots z_n)\in\mathbb{C}^n \mid 0<\Im(z_1)<\Im(z_2)<\cdots<\Im(z_n)<\beta\}, \tag{2.12}$$

and continuous on its closure $\bar{\mathcal{D}}^n_\beta\subset\mathbb{C}^n$. For every $n\in\mathbb{N}$, the "imaginary time" subset

$$\{(z_1,\ldots,z_n)\in\mathcal{D}^n_\beta \mid \Re(z_1)=\cdots=\Re(z_n)=0\}$$

is an inner set of uniqueness for functions analytic in $\mathcal{D}^n_\beta$ (see pages 101 and 352 of Ref. 77). Therefore, the Green functions (2.11), and hence the states (2.8), are completely determined by *the Matsubara functions*

$$\begin{aligned}\Gamma^{\Lambda}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n) &\stackrel{\mathrm{def}}{=} G^{\Lambda}_{F_1,\ldots,F_n}(\imath\tau_1,\ldots,\imath\tau_n)\\ &= \mathrm{trace}[F_1e^{-(\tau_2-\tau_1)H_\Lambda}F_2e^{-(\tau_3-\tau_2)H_\Lambda}\cdots F_ne^{-(\tau_{n+1}-\tau_n)H_\Lambda}]/\mathrm{trace}[e^{-\beta H_\Lambda}]\end{aligned} \tag{2.13}$$

taken at ordered arguments $0\le\tau_1\le\cdots\le\tau_n\le\tau_1+\beta\stackrel{\mathrm{def}}{=}\tau_{n+1}$, with all possible choices of $n\in\mathbb{N}$ and $F_1,\ldots,F_n\in C_{\mathrm{b}}(\mathbb{R}^{\nu|\Lambda|})$. Their extensions to $[0,\beta]^n$ are

$$\Gamma^{\Lambda}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n)=\Gamma^{\Lambda}_{F_{\sigma(1)},\ldots,F_{\sigma(n)}}\big(\tau_{\sigma(1)},\ldots,\tau_{\sigma(n)}\big),$$

where σ is the permutation of $\{1,2,\ldots,n\}$ such that $\tau_{\sigma(1)}\le\tau_{\sigma(2)}\le\cdots\le\tau_{\sigma(n)}$. One can show that for every $\theta\in[0,\beta]$,

$$\Gamma^{\Lambda}_{F_1,\ldots,F_n}(\tau_1+\theta,\ldots,\tau_n+\theta)=\Gamma^{\Lambda}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n), \tag{2.14}$$

where addition is modulo β. This periodicity along with the analyticity of the Green functions is equivalent to the KMS property of the state (2.8).

The central element of the Euclidean approach is the representation of the Matsubara functions (2.13) corresponding to $F_1,\ldots,F_n\in C_{\mathrm{b}}(\mathbb{R}^{\nu|\Lambda|})$ in the form of

$$\Gamma^{\Lambda}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n)=\int_{\Omega_\Lambda}F_1(\omega_\Lambda(\tau_1))\ldots F_n(\omega_\Lambda(\tau_n))\mu_\Lambda(\mathrm{d}\omega_\Lambda), \tag{2.15}$$

where μ_Λ is a certain probability measure on the space Ω_Λ, which we construct in the subsequent part of this section. This measure is called *a local Euclidean Gibbs measure*. By standard arguments, it is uniquely determined by the integrals (2.15). Since the Matsubara functions $\Gamma^{\Lambda}_{F_1,\ldots,F_n}$ uniquely determine the state ϱ_Λ, the representation (2.15) establishes a one-to-one correspondence between the local Gibbs states ϱ_Λ and local Euclidean Gibbs measures μ_Λ.

Thermodynamic properties of the model (1.1) are described by the Gibbs states corresponding to the whole set $\mathbb{L}$. Such states should be defined on the C^*-algebra of quasi-local observables $\mathfrak{C}$, being the norm-completion of the algebra of local observables $\cup_{\Lambda\Subset\mathbb{L}}\mathfrak{C}_\Lambda$. Here each $\mathfrak{C}_\Lambda$ is considered as a subalgebra of $\mathfrak{C}_{\Lambda'}$ for any Λ' containing Λ. The dynamics of the whole system is to be defined by the limits $\Lambda \nearrow \mathbb{L}$ of the time automorphisms (2.9), which would allow one to define the Gibbs states on $\mathfrak{C}$ as KMS states. This "algebraic" way can be realized for models described by bounded local Hamiltonians H_Λ, e.g., quantum spin models, see Sec. 6.2 of Ref. 23. For the model considered here, such limiting automorphisms do not exist and hence there is no canonical way to define Gibbs states of the whole infinite system. Therefore, the Euclidean approach based on the one-to-one correspondence between the local states and measures arising from the representation (2.15) seems to be the only way of developing a mathematical theory of the equilibrium thermodynamic properties of such models. For some versions of quantum crystals, a possibility of constructing the limiting states $\varrho = \lim_{\Lambda \nearrow \mathbb{L}} \varrho_\Lambda$ in terms of the limiting path measures $\mu = \lim_{\Lambda \nearrow \mathbb{L}} \mu_\Lambda$ was discussed in Refs. 15, 66, 67. The set of Euclidean Gibbs measures $\mathcal{G}^{\mathrm{t}}$ we construct and study in this article certainly includes all the limiting points of this type. Furthermore, there exist axiomatic methods, see Refs. 20, 35, analogous to the Osterwalder-Schrader reconstruction theory,[(37,79)] by means of which KMS states are constructed on certain von Neumann algebras from a complete set of Matsubara functions. In our case such a set consists of the functions

$$\Gamma^{\mu}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n) = \int_\Omega F_1(\omega(\tau_1))\cdots F_n(\omega(\tau_n))\mu(\mathrm{d}\omega), \quad \mu \in \mathcal{G}^{\mathrm{t}}, \tag{2.16}$$

corresponding to all local multiplication operators by bounded continuous functions $F_1, \ldots, F_n$. Therefore, the theory of Euclidean Gibbs measures presented in this article can be further developed towards constructing such algebras and states, which we leave as a task for the future.

2.3. Path Spaces and Local Euclidean Gibbs Measures

The local Euclidean Gibbs measures are defined on the spaces of continuous paths. These are continuous functions defined on the interval $[0, \beta]$, taking equal values at the endpoints (temperature loops). Here $\beta^{-1} = T > 0$ is absolute temperature. One can consider the loops as functions on the circle $S_\beta \cong [0, \beta]$ being a compact Riemannian manifold with Lebesgue measure $\mathrm{d}\tau$ and distance

$$|\tau - \tau'|_\beta \stackrel{\mathrm{def}}{=} \min\{|\tau - \tau'| \,;\, \beta - |\tau - \tau'|\}, \quad \tau, \tau' \in S_\beta. \tag{2.17}$$

As single-spin spaces we use the standard Banach spaces

$$C_\beta \stackrel{\mathrm{def}}{=} C(S_\beta \to \mathbb{R}^\nu), \qquad C^\sigma_\beta \stackrel{\mathrm{def}}{=} C^\sigma(S_\beta \to \mathbb{R}^\nu), \;\; \sigma \in (0,1),$$

of all continuous and Hölder-continuous functions $\omega_\ell : S_\beta \to \mathbb{R}^\nu$, equipped respectively with the supremum norm $|\omega_\ell|_{C_\beta}$ and with the Hölder norm

$$|\omega_\ell|_{C^\sigma_\beta} = |\omega_\ell|_{C_\beta} + \sup_{\tau,\tau' \in S_\beta,\ \tau \neq \tau'} \frac{|\omega_\ell(\tau) - \omega_\ell(\tau')|}{|\tau - \tau'|^\sigma_\beta}. \tag{2.18}$$

Along with them we use the real Hilbert space $L^2_\beta = L^2(S_\beta \to \mathbb{R}^\nu, \mathrm{d}\tau)$; its inner product and norm are denoted by $(\cdot,\cdot)_{L^2_\beta}$ and $|\cdot|_{L^2_\beta}$. By $\mathcal{B}(C_\beta)$, $\mathcal{B}(L^2_\beta)$ we denote the corresponding Borel σ-algebras. Then one defines dense continuous embeddings $C^\sigma_\beta \hookrightarrow C_\beta \hookrightarrow L^2_\beta$, that by the Kuratowski theorem, page 499 of Ref. 59, yields

$$C_\beta \in \mathcal{B}\big(L^2_\beta\big) \quad \text{and} \quad \mathcal{B}(C_\beta) = \mathcal{B}\big(L^2_\beta\big) \cap C_\beta. \tag{2.19}$$

The space of Hölder-continuous functions C^σ_β is not separable, however, as a subset of C_β or L^2_β, it is measurable (page 278 of Ref. 74). Given $\Lambda \subseteq \mathbb{L}$, we set

$$\Omega_\Lambda = \{\omega_\Lambda = (\omega_\ell)_{\ell\in\Lambda} \mid \omega_\ell \in C_\beta\}, \quad \Omega = \Omega_{\mathbb{L}} = \{\omega = (\omega_\ell)_{\ell\in\mathbb{L}} \mid \omega_\ell \in C_\beta\}. \tag{2.20}$$

These spaces are equipped with the product topology and with the Borel σ-algebras $\mathcal{B}(\Omega_\Lambda)$. Thereby, each Ω_Λ is a Polish space; its elements are called configurations in Λ. For $\Lambda \subset \Lambda'$, the decomposition $\omega_{\Lambda'} = \omega_\Lambda \times \omega_{\Lambda'\setminus\Lambda}$ defines an embedding $\Omega_\Lambda \hookrightarrow \Omega_{\Lambda'}$ by identifying $\omega_\Lambda \in \Omega_\Lambda$ with $\omega_\Lambda \times 0_{\Lambda'\setminus\Lambda} \in \Omega_{\Lambda'}$. By $\mathcal{P}(\Omega_\Lambda)$ and $\mathcal{P}(\Omega)$ we denote the sets of all probability measures on $(\Omega_\Lambda, \mathcal{B}(\Omega_\Lambda))$ and $(\Omega, \mathcal{B}(\Omega))$.

A ν-dimensional quantum harmonic oscillator of mass $m > 0$ and rigidity $a > 0$ is described by the Hamiltonian, c.f., (1.3),

$$H^{\mathrm{har}}_\ell = -\frac{1}{2m}\sum_{j=1}^{\nu}\left(\frac{\partial}{\partial x^{(j)}_\ell}\right)^2 + \frac{a}{2}|x_\ell|^2, \tag{2.21}$$

acting in the complex Hilbert space $L^2(\mathbb{R}^\nu)$. The operator semigroup $\exp(-\tau H^{\mathrm{har}}_\ell)$, $\tau \in [0,\beta]$, defines a Gaussian β-periodic Markov process – the periodic Ornstein-Uhlenbeck velocity process, see Ref. 45. In quantum statistical mechanics it first appeared in R. Høegh-Krohn's paper.[(41)] The canonical realization of this process on $(C_\beta, \mathcal{B}(C_\beta))$ is described by the path measure which one introduces as follows. In L^2_β, we define the self-adjoint (Laplace-Beltrami type) operator

$$A = \left(-m\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + a\right) \otimes \mathbf{I}, \tag{2.22}$$

where $\mathbf{I}$ is the identity operator in $\mathbb{R}^\nu$. Its spectrum consists of the eigenvalues

$$\lambda_k = m(2\pi k/\beta)^2 + a, \quad k \in \mathbb{Z}. \tag{2.23}$$

Thereby, the inverse A^{-1} is of trace class and the Fourier transform

$$\int_{L^2_\beta} \exp\left[\imath(\phi, \upsilon)_{L^2_\beta}\right]\chi(\mathrm{d}\upsilon) = \exp\left\{-\frac{1}{2}(A^{-1}\phi, \phi)_{L^2_\beta}\right\}, \quad \phi \in L^2_\beta \tag{2.24}$$

defines a Gaussian measure χ on $(L^2_\beta, \mathcal{B}(L^2_\beta))$. Employing the eigenvalues (2.23) one can show (by Kolmogorov's lemma, page 43 of Ref. 80) that

$$\chi\left(C^\sigma_\beta\right) = 1, \quad \text{for all } \sigma \in (0, 1/2). \tag{2.25}$$

Then $\chi(C_\beta) = 1$ and by (2.19) we redefine χ as a probability measure on $(C_\beta, \mathcal{B}(C_\beta))$. An account of the properties of χ may be found in Ref. 4. One of them, which plays a special role in our construction, follows directly from Fernique's theorem (see Theorem 1.3.24 in Ref. 26).

Proposition 2.3. *For every* $\sigma \in (0, 1/2)$*, there exists* $\lambda_\sigma > 0$ *such that*

$$\int_{L^2_\beta} \exp\left(\lambda_\sigma |\upsilon|^2_{C^\sigma_\beta}\right) \chi(\mathrm{d}\upsilon) < \infty. \tag{2.26}$$

The measure χ is the local Euclidean Gibbs measure for a single harmonic oscillator. The measure $\mu_\Lambda \in \mathcal{P}(\Omega_\Lambda)$ which corresponds to the system of interacting anharmonic oscillators located in $\Lambda \Subset \mathbb{L}$ is associated with a stationary β-periodic Markov process defined as follows. The marginal distributions of μ_Λ are given by the integral kernels of the operators $\exp(-\tau H_\Lambda)$, $\tau \in [0, \beta]$. This means that

$$\begin{aligned} &\mathrm{trace}\left[F_1 e^{-(\tau_2-\tau_1)H_\Lambda} F_2 e^{-(\tau_3-\tau_2)H_\Lambda} \cdots F_n e^{-(\tau_{n+1}-\tau_n)H_\Lambda}\right]/\mathrm{trace}[e^{-\beta H_\Lambda}] \\ &\quad = \int_{\Omega_\Lambda} F_1(\omega_\Lambda(\tau_1)\cdots F_n(\omega_\Lambda(\tau_n))\mu_\Lambda(\mathrm{d}\omega_\Lambda), \end{aligned} \tag{2.27}$$

for all $F_1, \ldots, F_n \in L^\infty(\mathbb{R}^{\nu|\Lambda|})$, $n \in \mathbb{N}$ and $\tau_1, \ldots, \tau_n \in S_\beta$, such that $\tau_1 \leq \cdots \leq \tau_n \leq \beta$, $\tau_{n+1} = \tau_1 + \beta$. And vice verse, the representation (2.27) uniquely, up to equivalence, defines H_Λ (see Ref. 44). By means of the Feynman-Kac formula the measure μ_Λ is obtained as a Gibbs modification

$$\mu_\Lambda(\mathrm{d}\omega_\Lambda) = \exp\left[-I_\Lambda(\omega_\Lambda)\right] \chi_\Lambda(\mathrm{d}\omega_\Lambda)/Z_\Lambda, \tag{2.28}$$

of the "free measure"

$$\chi_\Lambda(\mathrm{d}\omega_\Lambda) = \prod_{\ell\in\Lambda} \chi(\mathrm{d}\omega_\ell). \tag{2.29}$$

Here

$$I_\Lambda(\omega_\Lambda) = -\frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(\omega_\ell, \omega_{\ell'})_{L^2_\beta} + \sum_{\ell\in\Lambda}\int_0^\beta V_\ell(\omega_\ell(\tau))\,\mathrm{d}\tau \tag{2.30}$$

is the energy functional describing the system of interacting paths ω_ℓ, $\ell \in \Lambda$, whereas

$$Z_\Lambda = \int_{\Omega_\Lambda} \exp\left[-I_\Lambda(\omega_\Lambda)\right] \chi_\Lambda(\mathrm{d}\omega_\Lambda), \tag{2.31}$$

is the partition function. As mentioned above, μ_Λ is the local Gibbs measure, where *local* means corresponding to a $\Lambda \Subset \mathbb{L}$.

2.4. Tempered Configurations

The next step is to construct the equilibrium states of the whole infinite system (1.1). We are going to do this in the DLR approach, which is standard for classical (non-quantum) statistical mechanics, see Refs. 36, 73. In this approach, the Gibbs measures are constructed with the help of their local conditional distributions $\pi_\Lambda(\mathrm{d}\omega|\xi)$, $\Lambda \Subset \mathbb{L}$. These latter are defined by means of the energy functionals $I_\Lambda(\cdot|\xi)$ describing the interaction with a configuration $\xi \in \Omega$ fixed outside of Λ. In accordance with (2.2) it is

$$I_\Lambda(\omega|\xi) = I_\Lambda(\omega_\Lambda) - \sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} J_{\ell\ell'}(\omega_\ell, \xi_{\ell'})_{L^2_\beta}, \quad \omega \in \Omega, \tag{2.32}$$

where I_Λ is given by (2.30). Recall that $\omega = \omega_\Lambda \times \omega_{\Lambda^c}$; hence,

$$I_\Lambda(\omega|\xi) = I_\Lambda(\omega_\Lambda \times 0_{\Lambda^c}|0_\Lambda \times \xi_{\Lambda^c}). \tag{2.33}$$

Clearly, the second term in (2.32) makes sense for all $\xi \in \Omega$ only if the interaction has finite range. Otherwise, one has to restrict ξ to a subset of Ω, naturally defined by the condition

$$\forall \ell \in \mathbb{L}: \qquad \sum_{\ell'} |J_{\ell\ell'}| \cdot |(\omega_\ell, \xi_{\ell'})_{L^2_\beta}| < \infty, \tag{2.34}$$

that can be rewritten in terms of growth restrictions imposed on $\{|\xi_\ell|_{L^2_\beta}\}_{\ell\in\mathbb{L}}$, determined by the decay of $J_{\ell\ell'}$ (c.f., (2.5)). Configurations obeying such restrictions are called tempered. In one or another way tempered configurations always appear in the theory of system of unbounded spins, see Refs. 17, 24, 62, 69. To impose the restrictions we use special mappings, which define the scale of growth of $\{|\xi_\ell|_{L^2_\beta}\}_{\ell\in\mathbb{L}}$. Such mappings, called weights, are introduced by the following

Definition 2.4. Weights are the symmetric maps $w_\alpha : \mathbb{L} \times \mathbb{L} \to (0, +\infty)$, indexed by

$$\alpha \in \mathcal{I}=(\underline{\alpha}, \overline{\alpha}), \quad 0 \le \underline{\alpha} < \overline{\alpha} \le +\infty, \tag{2.35}$$

which satisfy the conditions:

(a) for any $\alpha \in \mathcal{I}$ and ℓ, $w_\alpha(\ell, \ell) = 1$;

(b) for any $\alpha \in \mathcal{I}$ and ℓ_1, ℓ_2, ℓ_3,

$$w_\alpha(\ell_1, \ell_2) \cdot w_\alpha(\ell_2, \ell_3) \le w_\alpha(\ell_1, \ell_3) \quad \textit{(triangle inequality)}, \tag{2.36}$$

(c) for any $\alpha, \alpha' \in \mathcal{I}$, such that $\alpha < \alpha'$, and arbitrary ℓ, ℓ',

$$w_{\alpha'}(\ell,\ell') \leq w_\alpha(\ell,\ell'), \qquad \lim_{|\ell-\ell'|\to+\infty} w_{\alpha'}(\ell,\ell')/w_\alpha(\ell,\ell') = 0. \tag{2.37}$$

The concrete choice of $\{w_\alpha\}_{\alpha\in\mathcal{I}}$ depends on the decay of $J_{\ell\ell'}$, which thus will be subject to the following

Assumption 2.5. *For all $\alpha \in \mathcal{I}$,*

$$\sup_\ell \sum_{\ell'} \log(1+|\ell-\ell'|)\cdot w_\alpha(\ell,\ell') < \infty; \tag{2.38}$$

$$\hat{J}_\alpha \stackrel{\mathrm{def}}{=} \sup_\ell \sum_{\ell'} |J_{\ell\ell'}|\cdot [w_\alpha(\ell,\ell')]^{-1} < \infty. \tag{2.39}$$

Given $\delta > 0$, which is a parameter of the theory, there exists $\alpha \in \mathcal{I}$, such that

$$\hat{J}_\alpha - \hat{J}_0 < \delta. \tag{2.40}$$

The choice of δ, based on the parameters of the model, will be done later. One observes that the conditions (2.38) and (2.39) are competitive. One can easily find examples of $J_{\ell\ell'}$ obeying (2.5), for which (2.38) and (2.39) cannot be satisfied simultaneously for any choice of the weights.

Let us give some typical examples. Suppose that

$$\sup_\ell \sum_{\ell'} |J_{\ell\ell'}|\cdot \exp\left(\alpha|\ell-\ell'|\right) < \infty, \qquad \text{for a certain } \alpha > 0. \tag{2.41}$$

The supremum of such α (possibly infinite) is denoted by $\overline{\alpha}$. Then we set

$$\mathcal{I} = (0,\overline{\alpha}), \qquad w_\alpha(\ell,\ell') = \exp\left(-\alpha|\ell-\ell'|\right). \tag{2.42}$$

If the condition (2.41) does not hold for any positive α, we assume that

$$\sup_\ell \sum_{\ell'} |J_{\ell\ell'}|\cdot \left(1+|\ell-\ell'|\right)^{\alpha d} < \infty, \tag{2.43}$$

for a certain $\alpha > 1$. Then $\overline{\alpha}$ is set to be the supremum of α obeying (2.43) and

$$\mathcal{I} = (1,\overline{\alpha}), \qquad w_\alpha(\ell,\ell') = \left(1+\varepsilon|\ell-\ell'|\right)^{-\alpha d}, \tag{2.44}$$

where the parameter $\varepsilon > 0$ will be chosen for (2.40) to be satisfied.

Given $u = (u_\ell)_{\ell\in\mathbb{L}} \in \mathbb{R}^{\mathbb{L}}$, ℓ_0, and $\alpha \in \mathcal{I}$, we set

$$|u|_{l^1(w_\alpha)} = \sum_\ell |u_\ell| w_\alpha(\ell_0,\ell), \qquad |u|_{l^\infty(w_\alpha)} = \sup_\ell \{|u_\ell| w_\alpha(\ell_0,\ell)\},$$

and introduce the Banach spaces

$$l^p(w_\alpha) = \left\{u \in \mathbb{R}^{\mathbb{L}} \;\middle|\; |u|_{l^p(w_\alpha)} < \infty\right\}, \quad p = 1, +\infty. \tag{2.45}$$

Remark 2.6. By (2.37), for $\alpha < \alpha'$, the embedding $l^1(w_\alpha) \hookrightarrow l^1(w_{\alpha'})$ is compact. By (2.39), for every $\alpha \in \mathcal{I}$, the operator $u \mapsto Ju$, defined as $(Ju)_\ell = \sum_{\ell'} J_{\ell\ell'} u_{\ell'}$, is bounded in both spaces $l^p(w_\alpha)$, $p = 1, +\infty$. Its norm does not exceed $\hat{J}_\alpha$.

For $\alpha \in \mathcal{I}$, we introduce

$$\Omega_\alpha = \left\{\omega \in \Omega \;\middle|\; \|\omega\|_\alpha \stackrel{\text{def}}{=} \left[\sum_\ell |\omega_\ell|^2_{L^2_\beta} w_\alpha(\ell_0, \ell)\right]^{1/2} < \infty\right\}, \tag{2.46}$$

and endow this set with the metric

$$\rho_\alpha(\omega, \omega') = \|\omega - \omega'\|_\alpha + \sum_\ell 2^{-|\ell|} \cdot \frac{|\omega_\ell - \omega'_\ell|_{C_\beta}}{1 + |\omega_\ell - \omega'_\ell|_{C_\beta}}, \tag{2.47}$$

which turns it into a Polish space.

Remark 2.7. The topology of each of the spaces $l^p(w_\alpha)$, Ω_α is independent of the particular choice of ℓ_0. This follows from the properties of the weights w_α assumed in Definition 2.4.

The set of tempered configurations is defined to be

$$\Omega^{\mathrm{t}} = \bigcap_{\alpha \in \mathcal{I}} \Omega_\alpha. \tag{2.48}$$

Equipped with the projective limit topology Ω^{t} becomes a Polish space as well. For any $\alpha \in \mathcal{I}$, we have continuous dense embeddings $\Omega^{\mathrm{t}} \hookrightarrow \Omega_\alpha \hookrightarrow \Omega$. Then by the Kuratowski theorem it follows that $\Omega_\alpha, \Omega^{\mathrm{t}} \in \mathcal{B}(\Omega)$ and the Borel σ-algebras of all these Polish spaces coincide with the ones induced on them by $\mathcal{B}(\Omega)$. Now we are at a position to complete the definition of the function (2.32).

Lemma 2.8. *For every $\alpha \in \mathcal{I}$ and $\Lambda \Subset \mathbb{L}$, the map $\Omega_\alpha \times \Omega_\alpha \ni (\omega, \xi) \mapsto I_\Lambda(\omega|\xi)$ is continuous. Furthermore, for every ball $B_\alpha(R) = \{\omega \in \Omega_\alpha \mid \rho_\alpha(0, \omega) < R\}$, $R > 0$, it follows that*

$$\inf_{\omega \in \Omega,\ \xi \in B_\alpha(R)} I_\Lambda(\omega|\xi) > -\infty, \qquad \sup_{\omega, \xi \in B_\alpha(R)} |I_\Lambda(\omega|\xi)| < +\infty. \tag{2.49}$$

Proof: As the functions $V_\ell : \mathbb{R}^\nu \to \mathbb{R}$ are continuous, the map $(\omega, \xi) \mapsto I_\Lambda(\omega_\Lambda)$ is continuous and bounded on the balls $B_\alpha(R)$. Furthermore,

$$\begin{aligned}
&\left| \sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} J_{\ell\ell'}(\omega_\ell, \xi_{\ell'})_{L^2_\beta} \right| \leq \sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} |J_{\ell\ell'}| \cdot |\omega_\ell|_{L^2_\beta} \cdot |\xi_{\ell'}|_{L^2_\beta} \\
&\quad = \sum_{\ell\in\Lambda} |\omega_\ell|_{L^2_\beta} [w_\alpha(0,\ell)]^{-1/2} \\
&\qquad \times \sum_{\ell'\in\Lambda^c} |J_{\ell\ell'}| \left[w_\alpha(0,\ell)/w_\alpha(0,\ell') \right]^{1/2} \cdot |\xi_{\ell'}|_{L^2_\beta} [w_\alpha(0,\ell')]^{1/2} \\
&\quad \leq \sum_{\ell\in\Lambda} |\omega_\ell|_{L^2_\beta} [w_\alpha(0,\ell)]^{-1/2} \sum_{\ell'\in\Lambda^c} |J_{\ell\ell'}| \cdot [w_\alpha(\ell,\ell')]^{-1/2} \cdot |\xi_{\ell'}|_{L^2_\beta} [w_\alpha(0,\ell')]^{1/2} \\
&\qquad \leq \hat{J}_\alpha \|\omega\|_\alpha \|\xi\|_\alpha \sum_{\ell\in\Lambda} [w_\alpha(0,\ell)]^{-1},
\end{aligned} \tag{2.50}$$

where we used the triangle inequality (2.36). This yields the continuity stated and the upper bound in (2.49). To prove the lower bound we employ the super-quadratic growth of V_ℓ assumed in (2.4). Then for any $\varkappa > 0$ and $\alpha \in \mathcal{I}$, one finds $C > 0$ such that for any $\omega \in \Omega$ and $\xi \in \Omega^{\mathrm{t}}$,

$$\begin{aligned}
I_\Lambda(\omega|\xi) &\geq B_V \beta |\Lambda| + A_V \beta^{1-r} \sum_{\ell\in\Lambda} |\omega_\ell|^{2r}_{L^2_\beta} - \frac{1}{2} \sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(\omega_\ell, \omega_{\ell'})_{L^2_\beta} \\
&\quad - \sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} J_{\ell\ell'}(\omega_\ell, \xi_{\ell'})_{L^2_\beta} \geq -C|\Lambda| + \varkappa \sum_{\ell\in\Lambda} |\omega_\ell|^2_{L^2_\beta} \\
&\quad - \hat{J}_\alpha \|\xi\|^2_\alpha \sum_{\ell\in\Lambda} w_\alpha(0,\ell).
\end{aligned} \tag{2.51}$$

To get the latter estimate we used the Minkowski inequality. □

Now for $\Lambda \Subset \mathbb{L}$ and $\xi \in \Omega^{\mathrm{t}}$, we introduce the partition function (c.f., (2.33))

$$Z_\Lambda(\xi) = \int_{\Omega_\Lambda} \exp\left[-I_\Lambda(\omega_\Lambda \times 0_{\Lambda^c}|\xi) \right] \chi_\Lambda(\mathrm{d}\omega_\Lambda). \tag{2.52}$$

An immediate corollary of the estimates (2.26) and (2.51) is the following

Proposition 2.9. *For every $\Lambda \Subset \mathbb{L}$, the function $\Omega^{\mathrm{t}} \ni \xi \mapsto Z_\Lambda(\xi) \in (0, +\infty)$ is continuous. Moreover, for any $R > 0$,*

$$\inf_{\xi\in B_\alpha(R)} Z_\Lambda(\xi) > 0, \qquad \sup_{\xi\in B_\alpha(R)} Z_\Lambda(\xi) < \infty. \tag{2.53}$$

2.5. Local Specification and Euclidean Gibbs Measures

We recall that the standard sources on the DLR approach are the books.[36,73] The local Gibbs specification is the family $\{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ of measure kernels

$$\mathcal{B}(\Omega) \times \Omega \ni (B, \xi) \mapsto \pi_\Lambda(B|\xi) \in [0, 1]$$

which we define as follows. For $\xi \in \Omega^{\mathrm{t}}$, $\Lambda \Subset \mathbb{L}$, and $B \in \mathcal{B}(\Omega)$, we set

$$\pi_\Lambda(B|\xi) = \frac{1}{Z_\Lambda(\xi)} \int_{\Omega_\Lambda} \exp\left[-I_\Lambda(\omega_\Lambda \times 0_{\Lambda^c}|\xi)\right] \mathbb{I}_B(\omega_\Lambda \times \xi_{\Lambda^c}) \chi_\Lambda(\mathrm{d}\omega_\Lambda), \quad (2.54)$$

where $\mathbb{I}_B$ stands for the indicator of B. We also set

$$\pi_\Lambda(\cdot|\xi) \equiv 0, \quad \text{for } \xi \in \Omega \setminus \Omega^{\mathrm{t}}. \quad (2.55)$$

To simplify notations we write $\pi_{\{\ell\}} = \pi_\ell$. From these definitions one readily derives a consistency property

$$\int_\Omega \pi_\Lambda(B|\omega)\pi_{\Lambda'}(\mathrm{d}\omega|\xi) = \pi_{\Lambda'}(B|\xi), \quad \Lambda \subset \Lambda', \quad (2.56)$$

which holds for all $B \in \mathcal{B}(\Omega)$ and $\xi \in \Omega$. Furthermore, by (2.51) it follows that for any $\xi \in \Omega$, $\sigma \in (0, 1/2)$, and $\varkappa > 0$,

$$\int_\Omega \exp\left\{\sum_{\ell \in \Lambda} \left(\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \varkappa |\omega_\ell|^2_{L^2_\beta}\right)\right\} \pi_\Lambda(\mathrm{d}\omega|\xi) < \infty, \quad (2.57)$$

where λ_σ is the same as in Proposition 2.3.

By $C_{\mathrm{b}}(\Omega_\alpha)$ (respectively, $C_{\mathrm{b}}(\Omega^{\mathrm{t}})$) we denote the Banach spaces of all bounded continuous functions $f : \Omega_\alpha \to \mathbb{R}$ (respectively, $f : \Omega^{\mathrm{t}} \to \mathbb{R}$) equipped with the supremum norm. For every $\alpha \in \mathcal{I}$, one has a natural embedding $C_{\mathrm{b}}(\Omega_\alpha) \hookrightarrow C_{\mathrm{b}}(\Omega^{\mathrm{t}})$.

Lemma 2.10. (Feller Property) *For every $\alpha \in \mathcal{I}$, $\Lambda \Subset \mathbb{L}$, and any $f \in C_{\mathrm{b}}(\Omega_\alpha)$, the function*

$$\Omega_\alpha \ni \xi \mapsto \pi_\Lambda(f|\xi)$$
$$\stackrel{\mathrm{def}}{=} \frac{1}{Z_\Lambda(\xi)} \int_{\Omega_\Lambda} f(\omega_\Lambda \times \xi_{\Lambda^c}) \exp\left[-I_\Lambda(\omega_\Lambda \times 0_{\Lambda^c}|\xi)\right] \chi_\Lambda(\mathrm{d}\omega_\Lambda), \quad (2.58)$$

belongs to $C_{\mathrm{b}}(\Omega_\alpha)$. The linear operator $f \mapsto \pi_\Lambda(f|\cdot)$ is a contraction on $C_{\mathrm{b}}(\Omega_\alpha)$.

Proof: By Lemma 2.8 and Proposition 2.9 the integrand

$$G^f_\Lambda(\omega_\Lambda|\xi) \stackrel{\mathrm{def}}{=} f(\omega_\Lambda \times \xi_{\Lambda^c}) \exp\left[-I_\Lambda(\omega_\Lambda \times 0_{\Lambda^c}|\xi)\right] / Z_\Lambda(\xi)$$

is continuous in both variables. Moreover, by (2.49) and (2.53) the map

$$\Omega_\alpha \ni \xi \mapsto \sup_{\omega_\Lambda \in \Omega_\Lambda} |G^f_\Lambda(\omega_\Lambda|\xi)|$$

is bounded on every ball $B_\alpha(R)$. This allows one to apply Lebesgue's dominated convergence theorem and obtain the continuity stated. Obviously,

$$\sup_{\xi \in \Omega_\alpha} |\pi_\Lambda(f|\xi)| \leq \sup_{\xi \in \Omega_\alpha} |f(\xi)|. \tag{2.59}$$

□

Note that by (2.54), for $\xi \in \Omega^{\rm t}$, $\alpha \in \mathcal{I}$, and $f \in C_{\rm b}(\Omega_\alpha)$,

$$\pi_\Lambda(f|\xi) = \int_\Omega f(\omega)\pi_\Lambda({\rm d}\omega|\xi). \tag{2.60}$$

Recall that the particular cases of our model were specified by Definition 2.2.. For $B \in \mathcal{B}(\Omega)$ and $U \in O(\nu)$, we set

$$U\omega = (U\omega_\ell)_{\ell\in\mathbb{L}} \qquad UB = \{U\omega \mid \omega \in B\}.$$

If $\mathbb{L}$ is a lattice, for a given ℓ_0, we set

$$t_{\ell_0}(\omega) = (\omega_{\ell-\ell_0})_{\ell\in\mathbb{L}}, \qquad t_{\ell_0}(B) = \{t_{\ell_0}(\omega) \mid \omega \in B\}.$$

Then if the model possesses the corresponding symmetry, one has

$$\pi_\Lambda(UB|U\xi) = \pi_\Lambda(B|\xi), \qquad \pi_{\Lambda+\ell}(t_\ell(B)|t_\ell(\xi)) = \pi_\Lambda(B|\xi), \tag{2.61}$$

which ought to hold for all U, ℓ, B, and ξ.

Definition 2.11. A measure $\mu \in \mathcal{P}(\Omega)$ is called a tempered Euclidean Gibbs measure if it satisfies the Dobrushin-Lanford-Ruelle (equilibrium) equation

$$\int_\Omega \pi_\Lambda(B|\omega)\mu({\rm d}\omega) = \mu(B), \quad \textit{for all} \quad \Lambda \Subset \mathbb{L} \ \textit{and} \ B \in \mathcal{B}(\Omega). \tag{2.62}$$

By $\mathcal{G}^{\rm t}$ we denote the set of all tempered Euclidean Gibbs measures of our model existing at a given β. So far we do not know whether $\mathcal{G}^{\rm t}$ is non-void; if it is, its elements should be supported by $\Omega^{\rm t}$. Indeed, by (2.54) and (2.55) $\pi_\Lambda(\Omega \setminus \Omega^{\rm t}|\xi) = 0$ for every $\Lambda \Subset \mathbb{L}$ and $\xi \in \Omega$. Then by (2.62),

$$\mu(\Omega \setminus \Omega^{\rm t}) = 0 \implies \mu(\Omega^{\rm t}) = 1. \tag{2.63}$$

Furthermore,

$$\mu\left(\left\{\omega \in \Omega^{\rm t} \mid \forall \ell \in \mathbb{L} : \ \omega_\ell \in C^\sigma_\beta\right\}\right) = 1, \tag{2.64}$$

which follows from (2.57). If the model is translation and/or rotation invariant, then, for every $U \in O(\nu)$ and $\ell \in \mathbb{L}$, the corresponding transformations preserve

$\mathcal{G}^{\mathrm{t}}$. That is, for any $\mu \in \mathcal{G}^{\mathrm{t}}$,

$$\Theta_U(\mu) \stackrel{\mathrm{def}}{=} \mu \circ U^{-1} \in \mathcal{G}^{\mathrm{t}}, \qquad \theta_\ell(\mu) \stackrel{\mathrm{def}}{=} \mu \circ t_\ell^{-1} \in \mathcal{G}^{\mathrm{t}}. \tag{2.65}$$

In particular, if $\mathcal{G}^{\mathrm{t}}$ is a singleton, its unique element should be invariant in the same sense as the model. One more invariance of the Euclidean Gibbs measures is connected with the dependence of their Matsubara functions on τ's.

Definition 2.12. A measure $\mu \in \mathcal{G}^{\mathrm{t}}$ is called τ-shift invariant if its Matsubara functions (2.16) have the property (2.14).

The τ-shift invariance is crucial for reconstructing quantum Gibbs states on von Neumann algebras, see Refs. 20, 35. This means that only the elements of $\mathcal{G}^{\mathrm{t}}$ which have this property are of physical relevance.

Given $\alpha \in \mathcal{I}$, by $\mathcal{W}_\alpha$ we denote the usual weak topology on the set of all probability measures $\mathcal{P}(\Omega_\alpha)$ defined by means of bounded continuous functions on Ω_α. By $\mathcal{W}^{\mathrm{t}}$ we denote the weak topology on $\mathcal{P}(\Omega^{\mathrm{t}})$. With these topologies the sets $\mathcal{P}(\Omega_\alpha)$ and $\mathcal{P}(\Omega^{\mathrm{t}})$ become Polish spaces (Theorem 6.5, page 46 of Ref. 70).

The proof of the existence of Euclidean Gibbs measures will be based on the following statement.

Lemma 2.13. *For each $\alpha \in \mathcal{I}$, every $\mathcal{W}_\alpha$-accumulation point $\mu \in \mathcal{P}(\Omega^{\mathrm{t}})$ of the family $\{\pi_\Lambda(\cdot|\xi) \mid \Lambda \Subset \mathbb{L},\ \xi \in \Omega^{\mathrm{t}}\}$ is an element of $\mathcal{G}^{\mathrm{t}}$.*

Proof: For each $\alpha \in \mathcal{I}$, $C_{\mathrm{b}}(\Omega_\alpha)$ is a measure defining class for $\mathcal{P}(\Omega^{\mathrm{t}})$. Then a measure $\mu \in \mathcal{P}(\Omega^{\mathrm{t}})$ solves (2.62) if and only if for any $f \in C_{\mathrm{b}}(\Omega_\alpha)$ and all $\Lambda \Subset \mathbb{L}$,

$$\int_{\Omega^{\mathrm{t}}} f(\omega)\mu(\mathrm{d}\omega) = \int_{\Omega^{\mathrm{t}}} \pi_\Lambda(f|\omega)\mu(\mathrm{d}\omega). \tag{2.66}$$

Let $\{\pi_{\Lambda_k}(\cdot|\xi_k)\}_{k\in\mathbb{N}}$ converge in $\mathcal{W}_\alpha$ to some $\mu \in \mathcal{P}(\Omega^{\mathrm{t}})$. For every $\Lambda \Subset \mathbb{L}$, one finds $k_\Lambda \in \mathbb{N}$ such that $\Lambda \subset \Lambda_k$ for all $k > k_\Lambda$. Then by (2.56), one has

$$\int_{\Omega^{\mathrm{t}}} f(\omega)\pi_{\Lambda_k}(\mathrm{d}\omega|\xi_k) = \int_{\Omega^{\mathrm{t}}} \pi_\Lambda(f|\omega)\pi_{\Lambda_k}(\mathrm{d}\omega|\xi_k).$$

Now by Lemma 2.10, one can pass to the limit $k \to +\infty$ and get (2.66). □

Let us stress that in the lemma above we suppose that the accumulation point is a probability measure on Ω^{t}. In general, the convergence of $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(\Omega^{\mathrm{t}})$ in every $\mathcal{W}_\alpha$, $\alpha \in \mathcal{I}$, does not yet imply its $\mathcal{W}^{\mathrm{t}}$-convergence. However, in Lemma 4.5 and Corollary 5.1 below we show that the topologies induced by $\mathcal{W}_\alpha$ and $\mathcal{W}^{\mathrm{t}}$ on a subset of $\mathcal{P}(\Omega)$, which includes $\mathcal{G}^{\mathrm{t}}$ and all $\pi_\Lambda(\cdot|\xi)$, coincide.

3. THE RESULTS

In the first subsection below we present the statements describing the general case, whereas the second subsection is dedicated to the case of $\nu = 1$ and $J_{\ell\ell'} \geq 0$.

3.1. Euclidean Gibbs Measures in the General Case

We begin by establishing existence of tempered Euclidean Gibbs measures and compactness of their set $\mathcal{G}^{\mathrm{t}}$. For models with non-compact spins, here they are even infinite-dimensional, such a property is far from being evident.

Theorem 3.1. *For every $\beta > 0$, the set of tempered Euclidean Gibbs measures $\mathcal{G}^{\mathrm{t}}$ is non-void and $\mathcal{W}^{\mathrm{t}}$- compact.*

The next theorem gives an exponential moment estimate similar to (2.26). Recall that the Hölder norm $|\cdot|_{C^\sigma_\beta}$ was defined by (2.18).

Theorem 3.2. *For every $\sigma \in (0, 1/2)$ and $\varkappa > 0$, there exists a positive constant $C_{3.1}$ such that, for any ℓ and for all $\mu \in \mathcal{G}^{\mathrm{t}}$,*

$$\int_\Omega \exp\left(\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \varkappa |\omega_\ell|^2_{L^2_\beta}\right)\mu(\mathrm{d}\omega) \leq C_{3.1}, \tag{3.1}$$

where λ_σ is the same as in (2.26).

According to (3.1), the one-site projections of each $\mu \in \mathcal{G}^{\mathrm{t}}$ are sub-Gaussian. The bound $C_{3.1}$ does not depend on ℓ and is the same for all $\mu \in \mathcal{G}^{\mathrm{t}}$, though it may depend on σ and $\varkappa$. The estimate (3.1) plays a crucial role in the theory of the set $\mathcal{G}^{\mathrm{t}}$. Such estimates are also important in the study of the Dirichlet operators H_μ associated with the measures $\mu \in \mathcal{G}^{\mathrm{t}}$, see Refs. 9, 10.

The set of tempered configurations Ω^{t} was introduced in (2.46), (2.48) by means of rather slack restrictions (c.f., (2.34)) imposed on the L^2_β-norms of ω_ℓ. By construction, the elements of $\mathcal{G}^{\mathrm{t}}$ are supported by this set, see (2.63). It turns out that they have a much smaller support (a kind of the Lebowitz-Presutti one, see Ref. 62). Given $b > 0$ and $\sigma \in (0, 1/2)$, we set

$$\begin{aligned}\Xi(b, \sigma) = \{\xi \in \Omega \mid (\forall \ell_0 \in \mathbb{L})\ (\exists \Lambda_{\xi,\ell_0} \Subset \mathbb{L})\ (\forall \ell \in \Lambda^c_{\xi,\ell_0}) :\\ |\xi_\ell|^2_{C^\sigma_\beta} \leq b \log(1 + |\ell - \ell_0|)\},\end{aligned} \tag{3.2}$$

which in view of (2.38) is a Borel subset of Ω^{t}.

Theorem 3.3. *For every $\sigma \in (0, 1/2)$, there exists $b > 0$, which depends on σ and on the parameters of the model only, such that for all $\mu \in \mathcal{G}^{\mathrm{t}}$,*

$$\mu(\Xi(b, \sigma)) = 1. \tag{3.3}$$

The last result in this group is a sufficient condition for $\mathcal{G}^{\mathrm{t}}$ to be a singleton, which holds for high temperatures (small β). It is obtained by controlling the "non-convexity" of the potential energy (2.3). Let us decompose

$$V_\ell = V_{1,\ell} + V_{2,\ell}, \tag{3.4}$$

where $V_{1,\ell} \in C^2(\mathbb{R}^\nu)$ is such that

$$-a \leq b \stackrel{\mathrm{def}}{=} \inf_{\ell} \inf_{x,y \in \mathbb{R}^\nu,\ y \neq 0} (V''_{1,\ell}(x)y, y)/|y|^2 < \infty. \tag{3.5}$$

As for the second term, we set

$$0 \leq \delta \stackrel{\mathrm{def}}{=} \sup_{\ell} \left\{ \sup_{x \in \mathbb{R}^\nu} V_{2,\ell}(x) - \inf_{x \in \mathbb{R}^\nu} V_{2,\ell}(x) \right\} \leq \infty. \tag{3.6}$$

Its role is to produce multiple minima of the potential energy responsible for eventual phase transitions. Clearly, the decomposition (3.4) is not unique; its optimal realizations for certain types of V_ℓ are discussed in section 6 of Ref. 11.

Theorem 3.4. *The set $\mathcal{G}^{\mathrm{t}}$ is a singleton if*

$$e^{\beta\delta} < (a + b)/\hat{J}_0. \tag{3.7}$$

Remark 3.5. The latter condition surely holds at all β if

$$\delta = 0 \quad \text{and} \quad \hat{J}_0 < a + b. \tag{3.8}$$

In this case the potential energy W_Λ given by (2.3) is convex. If the oscillators are harmonic, $\delta = b = 0$, which yields the stability condition

$$\hat{J}_0 < a. \tag{3.9}$$

The condition (3.7) does not contain the particle mass m; hence, the property stated holds also in the quasi-classical limit[4] $m \to +\infty$.

3.2. Ferroelectric Scalar Models

Recall that here we consider the case where $J_{\ell\ell'} \geq 0$ and $\nu = 1$.

[4] More details on this limit can be found in Ref. 4.

Let us introduce an order on the set $\mathcal{G}^{\mathrm{t}}$. As the components of the configurations $\omega \in \Omega$ are continuous functions $\omega_\ell : S_\beta \to \mathbb{R}^\nu$, we can set $\omega \leq \tilde{\omega}$ if $\omega_\ell(\tau) \leq \tilde{\omega}_\ell(\tau)$ for all ℓ and τ. Thereby, we define the following set of increasing functions

$$K_+(\Omega^{\mathrm{t}}) = \{f \in C_{\mathrm{b}}(\Omega^{\mathrm{t}}) \mid f(\omega) \leq f(\tilde{\omega}), \quad \text{if} \quad \omega \leq \tilde{\omega}\}, \tag{3.10}$$

which is a proper cone.

Lemma 3.6. *If for given $\mu, \tilde{\mu} \in \mathcal{G}^{\mathrm{t}}$, one has*

$$\mu(f) = \tilde{\mu}(f), \qquad \text{for all} \ \ f \in K_+(\Omega^{\mathrm{t}}), \tag{3.11}$$

then $\mu = \tilde{\mu}$.

The proof of this lemma will be given below in Sec. 6. We use it to establish the so called stochastic order on $\mathcal{G}^{\mathrm{t}}$.

Definition 3.7. For $\mu, \tilde{\mu} \in \mathcal{G}^{\mathrm{t}}$, we say that $\mu \leq \tilde{\mu}$, if

$$\mu(f) \leq \tilde{\mu}(f), \qquad \text{for all} \ \ f \in K_+(\Omega^{\mathrm{t}}). \tag{3.12}$$

Our first result in this subsection is the following

Theorem 3.8. *The set $\mathcal{G}^{\mathrm{t}}$ possesses maximal μ_+ and minimal μ_- elements in the sense of Definition 3.7. These elements are extreme and τ-shift invariant; they are also translation invariant if the model is translation invariant. If $V_\ell(-x) = V_\ell(x)$ for all ℓ, then $\mu_+(B) = \mu_-(-B)$ for all $B \in \mathcal{B}(\Omega)$.*

Now let the model be translation invariant, which in particular means $\mathbb{L} = \mathbb{Z}^d$. We are going to study the limiting pressure which contains important information about the thermodynamic properties of the model. A special attention will be paid to the dependence of the pressure on the external field h, c.f., (2.6). The corresponding analytic properties will be used in the study of phase transitions.

For $\Lambda \Subset \mathbb{L}$, we set

$$p_\Lambda(h, \xi) = \frac{1}{|\Lambda|} \log Z_\Lambda(\xi), \quad \ \xi \in \Omega^{\mathrm{t}}. \tag{3.13}$$

To simplify notations we write $p_\Lambda(h) = p_\Lambda(h, 0)$. For $\mu \in \mathcal{G}^{\mathrm{t}}$, we set

$$p_\Lambda^\mu(h) = \int_\Omega p_\Lambda(h, \xi)\mu(\mathrm{d}\xi). \tag{3.14}$$

If for a cofinal sequence $\mathcal{L}$, the limit

$$p^\mu(h) \stackrel{\mathrm{def}}{=} \lim_{\mathcal{L}} p_\Lambda^\mu(h) \tag{3.15}$$

exists, we shall call it pressure in the state μ. We shall also consider

$$p(h) \stackrel{\text{def}}{=} \lim_{\mathcal{L}} p_\Lambda(h). \tag{3.16}$$

To obtain these limits we impose a certain condition on the sequences $\mathcal{L}$. Given $l = (l_1, \ldots l_d)$, $l' = (l'_1, \ldots l'_d) \in \mathbb{L} = \mathbb{Z}^d$, such that $l_j < l'_j$ for all $j = 1, \ldots, d$, we set

$$\Gamma = \{\ell \in \mathbb{L} \mid l_j \leq \ell_j \leq l'_j, \quad \text{for all } j = 1, \ldots, d\}. \tag{3.17}$$

For this parallelepiped, let $\mathfrak{G}(\Gamma)$ be the family of all pair-wise disjoint translates of Γ which cover $\mathbb{L}$. Then for $\Lambda \Subset \mathbb{L}$, we set $N_-(\Lambda|\Gamma)$ (respectively, $N_+(\Lambda|\Gamma)$) to be the number of the elements of $\mathfrak{G}(\Gamma)$ which are contained in Λ (respectively, have non-void intersections with Λ). Then we introduce, see Ref. 75,

Definition 3.9. A cofinal sequence $\mathcal{L}$ is a van Hove sequence if for every Γ,

$$(a) \ \lim_{\mathcal{L}} N_-(\Lambda|\Gamma) = +\infty; \qquad (b) \ \lim_{\mathcal{L}} (N_-(\Lambda|\Gamma)/N_+(\Lambda|\Gamma)) = 1. \tag{3.18}$$

Theorem 3.10. *For every $h \in \mathbb{R}$ and any van Hove sequence $\mathcal{L}$, the limits (3.15) and (3.16) exist, do not depend on the particular choice of $\mathcal{L}$, and are equal, that is $p(h) = p^\mu(h)$ for each $\mu \in \mathcal{G}^{\mathrm{t}}$.*

The following result, which will be proven in Sec. 7 below, is a consequence of Theorems 3.10 and 3.8.

Corollary 3.11. *If $p(h)$ is differentiable at a given $h \in \mathbb{R}$, then $\mathcal{G}^{\mathrm{t}}$ is a singleton at this h.*

In the DLR approach the multiplicity of Gibbs states corresponds to phase transitions. In physical systems structural phase transitions manifest themselves in the macroscopic displacements of particles from their equilibrium positions. For translation invariant ferroelectric models with $V_\ell = V$ obeying certain conditions, the appearance of such macroscopic displacements at low temperatures was proven in Refs. 16, 27, 39, 48, 71. Thus, one can expect that $|\mathcal{G}^{\mathrm{t}}| > 1$ at big β. The latter fact would readily imply the appearance of macroscopic displacements, but the converse need not to be true in general. To avoid technical complications we prove this for $\mathbb{L} = \mathbb{Z}^d$, $d \geq 3$ – by means of correlation inequalities this result can be extended to the case of irregular $\mathbb{L} \subset \mathbb{R}^d$.

Let us impose further conditions on $J_{\ell\ell'}$ and V_ℓ. The first one is

$$\inf_{\ell,\ell' :\ |\ell-\ell'|=1} J_{\ell\ell'} \stackrel{\text{def}}{=} J > 0. \tag{3.19}$$

Next we suppose that V_ℓ are even continuous functions and the upper bound in (2.4) can be chosen in the form

$$V(x_\ell) = \sum_{s=1}^{r} b^{(s)} x_\ell^{2s}; \quad 2b^{(1)} < -a; \quad b^{(s)} \geq 0, \quad s \geq 2, \tag{3.20}$$

where a is the same as in (2.21) or in (2.3), and $r \geq 2$ is either a positive integer or $r = +\infty$. In the latter case we assume that the series

$$\Phi(t) = \sum_{s=2}^{+\infty} \frac{(2s)!}{2^{s-1}(s-1)!} b^{(s)} t^{s-1}, \tag{3.21}$$

converges at some $t > 0$. Since $2b^{(1)} + a < 0$, the equation

$$a + 2b^{(1)} + \Phi(t) = 0, \tag{3.22}$$

has a unique solution $t_* > 0$. Finally, we suppose that for every ℓ,

$$V(x_\ell) - V_\ell(x_\ell) \leq V(\tilde{x}_\ell) - V_\ell(\tilde{x}_\ell), \quad \text{whenever } x_\ell^2 \leq \tilde{x}_\ell^2. \tag{3.23}$$

If $V_\ell(x_\ell) = v_\ell(x_\ell^2)$ and v_ℓ are differentiable, the condition (3.23) may be formulated as an upper bound for v_ℓ'. For $d \geq 3$, we set

$$\theta_d = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi]^d} \frac{\mathrm{d}p}{E(p)}, \qquad E(p) = \sum_{j=1}^{d} [1 - \cos p_j]. \tag{3.24}$$

Let also $f : [0, +\infty) \to [0, 1)$ be the function defined implicitly by

$$f(t \tanh t) = t^{-1} \cdot \tanh t, \quad \text{for } t > 0, \text{ and } f(0) = 1. \tag{3.25}$$

It is convex and monotone decreasing on $(0, +\infty)$. For an account of its properties see Ref. 29, where it was introduced.

By (3.25) one readily proves that for every fixed $\alpha > 0$, the function

$$(0, +\infty) \ni t \mapsto \phi(t, \alpha) = \alpha t f(t/\alpha), \tag{3.26}$$

is monotone increasing to α^2 as $t \to +\infty$.

Theorem 3.12. *Let $d \geq 3$ and the above assumptions hold. Then under the condition*

$$J > \theta_d / 8 m t_*^2, \tag{3.27}$$

there exists $\beta_* > 0$ such that $|\mathcal{G}^{\mathrm{t}}| > 1$ whenever $\beta > \beta_*$. The bound β_* is the unique solution of the equation

$$2\theta_d m / J = \phi(\beta, 4 m t_*). \tag{3.28}$$

As was shown in Refs. 2, 6, 50, quantum effects, occurring in particular at small values of the particle mass m, can suppress abnormal fluctuations. Thus, one might expect that such effects can cause $|\mathcal{G}^{\rm t}| = 1$ occurring at all temperatures. The strongest result in this domain—the uniqueness at all β due to quantum effects for the model with nearest neighbor interaction and a certain type of V (so called EMN, see Ref. 31)—was proven in Ref. 5. In Theorem 3.13 below we extend this result in two directions. We consider a substantially larger class of anharmonic potentials and make precise the bounds of the uniqueness regime. Furthermore, unlike to the mentioned papers, we do not suppose that the interaction has finite range and that $\mathbb{L}$ is regular. Regarding the anharmonic potentials we suppose that each V_ℓ is even and hence can be written

$$V_\ell(x) = v_\ell(x^2). \tag{3.29}$$

Furthermore, we suppose that there exists the function $v : [0, +\infty) \to \mathbb{R}$ which is convex and such that

$$v_\ell(t) - v(t) \leq v_\ell(\theta) - v(\theta) \quad \text{whenever} \;\; t < \theta. \tag{3.30}$$

In typical cases of V_ℓ, like (2.6), as such a v one can take a convex polynomial of degree $r \geq 2$.

Next we introduce the following one-particle Hamiltonian (c.f., (2.21), (2.2))

$$\tilde{H} = -\frac{1}{2m}\left(\frac{\partial}{\partial x}\right)^2 + \frac{a}{2}x^2 + v(x^2), \quad x \in \mathbb{R}. \tag{3.31}$$

It has purely discrete non-degenerate spectrum $\{E_n\}_{n\in\mathbb{N}_0}$. Thus, one can define the parameter

$$\Delta = \min_{n\in\mathbb{N}} (E_n - E_{n-1}), \tag{3.32}$$

which is positive and depends on the model parameters m, a, and on the choice of v. Recall, that $\hat{J}_0$ was defined by (2.5).

Theorem 3.13. *Let the anharmonic potentials V_ℓ be as above. Then the set of Euclidean Gibbs measures is a singleton if*

$$m\Delta^2 > \hat{J}_0. \tag{3.33}$$

Note that the above result is independent of $\beta > 0$ and that (3.33) is a stability condition like (3.8), where the parameter $m\Delta^2$ appears as the oscillator rigidity. If it holds, a stability-due-to-quantum-effects occurs, c.f., Refs. 6, 49, 50, 54. If v is a polynomial of degree $r \geq 2$, the rigidity $m\Delta^2$ is a continuous function of the particle mass m; it gets small in the quasi-classical limit $m \to +\infty$, see

Ref. 54. At the same time, for $m \to 0+$, one has $m\Delta^2 = O(m^{-(r-1)/(r+1)})$, see Refs. 2, 54. Hence, (3.33) certainly holds in the small mass limit, c.f., Refs 3, 5. To compare the latter statement with Theorem 3.12 let us assume that $\mathbb{L} = \mathbb{Z}^d$, $d \geq 3$, $J_{\ell\ell'} = J$ iff $|\ell - \ell'| = 1$, and all V_ℓ coincide with the function given by (3.20). Then the parameter (3.32) obeys the estimate $\Delta < 1/2mt_*$, see Ref. 54, where t_* is the same as in (3.27), (3.28). In this case the condition (3.33) can be rewritten as

$$J < 1/8dmt_*^2. \tag{3.34}$$

One can show that $\theta_d > 1/d$ and $d\theta_d \to 1$ as $d \to +\infty$; hence, the estimates (3.27) and (3.34), which give sufficient conditions for the phase transition to occur or to be suppressed, become asymptotically sharp.

Consider again a translation invariant version of our model, i.e., $\mathbb{L} = \mathbb{Z}^d$. Set

$$\mathcal{F}_{\text{Laguerre}} = \left\{ \varphi : \mathbb{R} \to \mathbb{R} \;\middle|\; \varphi(t) = \varphi_0 \exp(\gamma_0 t) t^n \prod_{i=1}^{\infty} (1 + \gamma_i t) \right\}, \tag{3.35}$$

where $\varphi_0 > 0, n \in \mathbb{N}_0, \gamma_i \geq 0$ for all $i \in \mathbb{N}_0$, and $\sum_{i=1}^{\infty} \gamma_i < \infty$. Each $\varphi \in \mathcal{F}_{\text{Laguerre}}$ can be extended to an entire function $\varphi : \mathbb{C} \to \mathbb{C}$, which has no zeros outside of $(-\infty, 0]$. These are Laguerre entire functions, see Refs. 42, 52, 57. In the next theorem the parameter a is the same as in (2.21).

Theorem 3.14. *Let the model we consider be translation invariant and the anharmonic potential be of the form*

$$V(x) = v(x^2) - hx, \quad h \in \mathbb{R}, \tag{3.36}$$

where $v(0) = 0$ and is such that for a certain $b \geq -a/2$, the derivative v' obeys the condition $b + v' \in \mathcal{F}_{\text{Laguerre}}$. Then the set $\mathcal{G}^{\text{t}}$ is a singleton if $h \neq 0$.

3.3. Comments

In what follows, we have developed a consistent rigorous theory of the equilibrium thermodynamic properties of quantum models like (1.1), based on a path measure representation of local Gibbs states (2.8). In this theory, the model is interpreted as a system of infinite-dimensional spins; its global properties are described by the Euclidean Gibbs measures constructed with the help of the DLR equation. As the spins are infinite-dimensional, the methods employed are more involved and complicated than those used for classical models. Additional complications arise from the fact that we study a general case, where the model has no spacial regularity and the interaction is of infinite range. In view of the latter possibility, the only way to develop the theory is to impose a priori restrictions on the support of the Gibbs measures, which was done by means of the weights obeying the conditions (2.36)–(2.39). These conditions are competitive and can contradict each other if the interaction decays too slowly. If they are satisfied, the set of tempered Gibbs measures $\mathcal{G}^{\mathrm{t}}$ is non-void, Theorem 3.1. A posteriori, by Theorem 3.3 its elements have much smaller support than Ω^{t}, which does not depend on the particular choice of the weights. If the interaction has finite range, the local specification and the corresponding Gibbs measures can be defined with no support restrictions as probability measures on Ω. The existence of Gibbs measures would follow from the proof of Theorem 3.1. However, in this case the set of all Gibbs measures would be too big—it may contain "improper" elements, which have no physical meaning and hence should be excluded from the theory. This can be performed by means of the weights satisfying the same conditions, except for (2.39) which now holds automatically. Once this is done, the corresponding tempered Gibbs measures obey the estimate (3.1) and hence have the support described by Theorem 3.3, independent of the weights.

Now let us compare our results with those known for similar classical and quantum models.

- **Theorem 3.1.** A standard tool for proving the existence of Gibbs measures is the celebrated Dobrushin criterion, see Theorem 1 in Ref. 25. To apply it in our case one should find a compact positive function h defined on C_β such that for all ℓ and $\xi \in \Omega$,

$$\int_\Omega h(\omega_\ell)\pi_\ell(\mathrm{d}\omega|\xi) \leq A + \sum_{\ell' \neq \ell} I_{\ell\ell'} h(\xi_{\ell'}), \tag{3.37}$$

where

$$A > 0; \qquad I_{\ell\ell'} \geq 0 \quad \text{for all } \ell, \ell', \qquad \text{and} \qquad \sup_\ell \sum_{\ell'} I_{\ell\ell'} < 1.$$

Then (3.37) would yield that for any $\xi \in \Omega$, such that $\sup_\ell h(\xi_\ell) < \infty$, the family $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda \Subset \mathbb{L}}$ is relatively compact in the weak topology on $\mathcal{P}(\Omega)$ (but not yet in $\mathcal{W}_\alpha$, $\mathcal{W}^{\mathrm{t}}$). Next one would have to show that any accumulation point of $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda \Subset \mathbb{L}}$ is a Gibbs measure, which is much stronger than the fact established by our Lemma 2.13. Such a scheme was used in Refs. 17, 24, 82 where the existence of Gibbs measures for lattice systems with the single-spin space $\mathbb{R}$ was proven. In those papers the specific properties of the models, such as attractiveness and translation invariance, were cricial. The direct extension of this scheme to quantum models seems to be impossible. The scheme we employ for proving Theorem 3.1 is based on compactness arguments in the topologies $\mathcal{W}_\alpha, \mathcal{W}^{\mathrm{t}}$. After obvious modifications it can be applied to models with more general inter-particle interactions. Further comments on this item follow Corollary 4.2.

- **Theorem 3.2** gives a uniform exponential moment estimate for tempered Euclidean Gibbs measures in terms of model parameters, which in principle can be proven before establishing the existence. For systems of classical unbounded spins, the problem of deriving such estimates was first posed in Ref. 17 (see the discussion following Corollary 4.2). For quantum anharmonic systems, similar estimates were obtained in the so called analytic approach, alternative to the traditional DLR scheme, see Refs. 7, 8, 13. In this analytic approach, $\mathcal{G}^{\rm t}$ is defined as the set of probability measures satisfying an integration-by-parts formula, determined by the model. This gives additional tools for studying $\mathcal{G}^{\rm t}$ and provides a background for the stochastic dynamics method in which the Gibbs measures are treated as invariant distributions for certain infinite-dimensional stochastic evolution equations, see Ref. 14. In both analytic and stochastic dynamics methods one imposes a number of technical conditions on the interaction potentials and uses advanced tools of stochastic analysis. The method we employ for proving Theorem 3.2 is much more elementary. At the same time, Theorem 3.2 gives an improvement of the corresponding results of Ref. 7 because: (a) the estimate (3.1) gives a much stronger bound; (b) we do not assume that V_ℓ are differentiable—an important assumption of the analytic approach.
- **Theorem 3.3.** As might be clear from the proof of this theorem, every $\mu \in \mathcal{P}(\Omega^{\rm t})$ obeying the estimate (3.1) possesses the support property (3.3). For Gibbs measures of classical lattice systems of unbounded spins, a similar property was first established in Ref. 62; hence, one can call $\Xi(b, \sigma)$ a Lebowitz-Presutti type support. This result of Ref. 62 was obtained by means of Ruelle's superstability estimates,[76] applicable to translation invariant models only. Its generalization to translation invariant quantum model was done in Ref. 69, where superstable Gibbs measures were specified by the following support property

$$\sup_{N\in\mathbb{N}} \left\{ (1+2N)^{-d} \sum_{\ell:|\ell|\leq N} |\omega_\ell|^2_{L^2_\beta} \right\} \leq C(\omega), \quad \mu - \text{a.e.}.$$

Here we note that by the Birkhoff-Khinchine ergodic theorem, for any translation invariant measure $\mu \in \mathcal{P}(\Omega^{\rm t})$ obeying (3.1), it follows a much stronger support property—for every $\sigma \in (0, 1/2)$, $\varkappa > 0$, and μ-almost all ω,

$$\sup_{N\in\mathbb{N}} \left\{ (1+2N)^{-d} \sum_{\ell:|\ell|\leq N} \exp\left(\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \varkappa |\omega_\ell|^2_{L^2_\beta} \right) \right\} \leq C(\sigma, \varkappa, \omega).$$

In particular, every periodic Euclidean Gibbs measure constructed in subsection 7.5 below has this property.

- **Theorem 3.4** establishes a sufficient uniqueness condition, holding in particular at high-temperatures (small β). Here we follow the papers, [11,12] where a similar uniqueness statement was proven for translation invariant ferromagnetic scalar version of our model. This was done by means of another renown Dobrushin result, Theorem 4 in Ref. 25, which gives a sufficient condition for the uniqueness of Gibbs measures. The main tool used in Refs. 11, 12 for estimating the elements of the Dobrushin matrix was the logarithmic Sobolev inequality for the kernels π_ℓ.
- **Theorem 3.8.** For classical ferromagnetic spin models, similar results were obtained in Refs. 17, 73 and Ref. 60, 62 . The extreme elements $\mu_\pm$ play an important role in proving Theorems 3.12, 3.13, and 3.14.
- **Theorem 3.10.** For classical ferromagnetic spin models, a similar statement was proven in Refs. 17, 62.
- **Theorem 3.12.** For translation invariant lattice models, phase transitions are established by means of the infrared estimates, see Refs. 16, 27, 39, 48, 71. Here we use a version of the technique developed in those papers and the corresponding correlation inequalities which allow us to compare the model considered with its translation invariant version (reference model).
- **Theorem 3.13.** For translation invariant models with finite range interactions and with the anharmonic potential being the polynomial (2.6) with all $b^{(s)} \geq 0$ except for $b^{(1)}$ (the so called EMN-class, see Ref. 31), the uniqueness by quantum effects was proven in Ref. 5 (see also Ref. 3). With the help of the extreme elements $\mu_\pm \in \mathcal{G}^{\mathrm{t}}$ we essentially extend the results of those papers. As in the case of Theorem 3.12, we employ correlation inequalities to compare the model considered with a proper reference model.
- **Theorem 3.14.** For classical lattice models, the uniqueness at nonzero h was proven in [17,60,62] under the condition that the potential (3.36) possesses the property which we establish below in Definition 8.1. The novelty of Theorem 3.14 is that it describes a quantum model and gives an explicit sufficient condition for V to possess such a property[5] .This theorem is valid also in the quasi-classical limit $m \to +\infty$, in which it covers all the cases considered in Refs. 17, 60, 62. For $(\phi^4)_2$ Euclidean quantum fields, a similar statement was proven in Ref. 34.

[5] Examples follow Proposition 8.2.

4. PROPERTIES OF THE LOCAL GIBBS SPECIFICATION

4.1. Moment Estimates

Moment estimates for the kernels (2.54) we are going to derive will allow for proving the $\mathcal{W}^{\mathrm{t}}$-relative compactness of the set $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda\Subset\mathbb{L}}$, which by Lemma 2.13 will yield $\mathcal{G}^{\mathrm{t}} \neq \emptyset$. Integrating them over $\xi \in \Omega^{\mathrm{t}}$ we will get by the DLR Eq. (2.62) the corresponding estimates for the elements of $\mathcal{G}^{\mathrm{t}}$. Recall that π_ℓ stands for $\pi_{\{\ell\}}$.

Lemma 4.1. *For any $\varkappa$, $\vartheta > 0$, and $\sigma \in (0, 1/2)$, there exists $C_{4.1} > 0$ such that for all $\ell \in \mathbb{L}$ and $\xi \in \Omega^{\mathrm{t}}$,*

$$\int_\Omega \exp\left\{\lambda_\sigma|\omega_\ell|^2_{C^\sigma_\beta} + \varkappa|\omega_\ell|^2_{L^2_\beta}\right\}\pi_\ell(\mathrm{d}\omega|\xi) \leq \exp\left\{C_{4.1} + \vartheta\sum_{\ell'}|J_{\ell\ell'}|\cdot|\xi_{\ell'}|^2_{L^2_\beta}\right\}. \tag{4.1}$$

Here $\lambda_\sigma > 0$ is the same as in (3.1).

Proof: Note that by (2.57) the left-hand side of (4.1) is finite and the second term in $\exp\{\cdot\}$ on the right-hand side is also finite since $\xi \in \Omega^{\mathrm{t}}$.

For any $\vartheta > 0$, one has (see (2.5))

$$\left|\sum_{\ell'} J_{\ell\ell'}(\omega_\ell, \xi_{\ell'})_{L^2_\beta}\right| \leq \frac{\hat{J}_0}{2\vartheta}|\omega_\ell|^2_{L^2_\beta} + \frac{\vartheta}{2}\sum_{\ell'}|J_{\ell\ell'}|\cdot|\xi_{\ell'}|^2_{L^2_\beta}, \tag{4.2}$$

which holds for all $\omega, \xi \in \Omega^{\mathrm{t}}$. By these estimates and (2.30), (2.32), (2.52), (2.54)

$$\mathrm{LHS}(4.1) \leq [1/Y_\ell(\vartheta)]\cdot\exp\left\{\vartheta\sum_{\ell'}|J_{\ell\ell'}|\cdot|\xi_{\ell'}|^2_{L^2_\beta}\right\}$$

$$\times\int_\Omega \exp\left\{\lambda_\sigma|\omega_\ell|^2_{C^\sigma_\beta} + (\varkappa + \hat{J}_0/2\vartheta)|\omega_\ell|^2_{L^2_\beta} - \int_0^\beta V_\ell(\omega_\ell(\tau))\,\mathrm{d}\tau\right\}\chi(\mathrm{d}\omega_\ell), \tag{4.3}$$

where

$$Y_\ell(\vartheta) = \int_\Omega \exp\left\{-\frac{\hat{J}_0}{2\vartheta}\cdot|\omega_\ell|^2_{L^2_\beta} - \int_0^\beta V_\ell(\omega_\ell(\tau))\,\mathrm{d}\tau\right\}\chi(\mathrm{d}\omega_\ell).$$

Now we use the upper bound (2.4) to estimate $\inf_\ell Y_\ell(\vartheta)$, the lower bound (2.4) to estimate the integrand in (4.3), take into account Proposition 2.1, and arrive at (4.1). □

By Jensen's inequality we readily get from (4.1) the following Dobrushin-like bound.

Corollary 4.2. *For all ℓ and $\xi \in \Omega^{\mathrm{t}}$, the measures $\pi_\ell(\cdot|\xi)$, obey the estimate*

$$\int_\Omega h(\omega_\ell)\pi_\ell(\mathrm{d}\omega|\xi) \leq C_{4.1} + (\vartheta/\varkappa)\sum_{\ell'} |J_{\ell\ell'}| \cdot h(\xi_{\ell'}), \tag{4.4}$$

with

$$h(\omega_\ell) = \lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \varkappa|\omega_\ell|^2_{L^2_\beta}, \tag{4.5}$$

which is a compact function $h : C_\beta \to \mathbb{R}$.

For translation invariant lattice systems with the single-spin space $\mathbb{R}$ and ferromagnetic pair interactions, integrability estimates like

$$\log\left\{\int_{\mathbb{R}^{\mathbb{L}}} \exp(\lambda|x_\ell|)\pi_\ell(\mathrm{d}x|y)\right\} < A + \sum_{\ell'} I_{\ell\ell'}|y_{\ell'}|,$$

were first obtained by J. Bellissard and R. Høegh-Krohn, see Proposition III.1 and Theorem III.2 in Ref. 17. Dobrushin type estimates like (3.37) were also proven in Refs. 24, 82. The methods used there essentially employed the properties of the model and hence cannot be of use in our situation. Our method of getting such estimates is much simpler; at the same time, it is applicable in both cases—classical and quantum. Its peculiarities are: (a) first we prove the exponential integrability (4.1) and then derive the Dobrushin bound (4.4) rather than prove it directly; (b) the function (4.5) consists of two additive terms, the first of which is to guarantee the compactness while the second one controls the inter-particle interaction.

Now by means of (4.1) we obtain the corresponding estimates for the kernels π_Λ with arbitrary $\Lambda \Subset \mathbb{L}$. Let the parameters σ, $\varkappa$, and λ_σ be the same as in (4.1). For $\ell \in \Lambda \Subset \mathbb{L}$, we define

$$n_\ell(\Lambda|\xi) = \log\left\{\int_\Omega \exp\left(\lambda_\sigma|\omega_\ell|^2_{C^\sigma_\beta} + \varkappa|\omega_\ell|^2_{L^2_\beta}\right)\pi_\Lambda(\mathrm{d}\omega|\xi)\right\}, \tag{4.6}$$

which is finite by (2.57).

Lemma 4.3. *For every $\alpha \in \mathcal{I}$, there exists $C_{4.7}(\alpha) > 0$ such that for all $\xi \in \Omega^{\mathrm{t}}$,*

$$\limsup_{\Lambda \nearrow \mathbb{L}} \sum_{\ell\in\Lambda} n_\ell(\Lambda|\xi) w_\alpha(\ell_0, \ell) \leq C_{4.7}(\alpha); \tag{4.7}$$

hence,

$$\limsup_{\Lambda\nearrow\mathbb{L}} n_{\ell_0}(\Lambda|\xi) \leq C_{4.7}(\alpha), \quad for \ \ any \ \ \ell_0. \tag{4.8}$$

Thereby, there exists $C_{4.9}(\ell, \xi) > 0$ such that for all $\Lambda \Subset \mathbb{L}$ containing ℓ,

$$n_\ell(\Lambda|\xi) \leq C_{4.9}(\ell, \xi). \tag{4.9}$$

Proof: Given $\varkappa > 0$ and $\alpha \in \mathcal{I}$, we fix $\vartheta > 0$ such that

$$\vartheta \sum_{\ell'} |J_{\ell\ell'}| \leq \vartheta \hat{J}_0 \leq \vartheta \hat{J}_\alpha < \varkappa. \tag{4.10}$$

Then integrating both sides of the bound (4.1) with respect to the measure $\pi_\Lambda(\mathrm{d}\omega|\xi)$ we get

$$\begin{aligned} n_\ell(\Lambda|\xi) &\leq C_{4.1} + \vartheta \sum_{\ell' \in \Lambda^c} |J_{\ell\ell'}| \cdot |\xi_{\ell'}|^2_{L^2_\beta} \\ &\quad + \log \left\{ \int_\Omega \exp\left(\vartheta \sum_{\ell' \in \Lambda} |J_{\ell\ell'}| \cdot |\omega_{\ell'}|^2_{L^2_\beta} \right) \pi_\Lambda(\mathrm{d}\omega|\xi) \right\} \\ &\leq C_{4.1} + \vartheta \sum_{\ell' \in \Lambda^c} |J_{\ell\ell'}| \cdot |\xi_{\ell'}|^2_{L^2_\beta} + \vartheta/\varkappa \sum_{\ell' \in \Lambda} |J_{\ell\ell'}| \cdot n_{\ell'}(\Lambda|\xi). \end{aligned} \tag{4.11}$$

Here we have used (4.10) and the multiple Hölder inequality

$$\int \left(\prod_{i=1}^n \varphi_i^{\alpha_i} \right) \mathrm{d}\mu \leq \prod_{i=1}^n \left(\int \varphi_i \mathrm{d}\mu \right)^{\alpha_i},$$

in which μ is a probability measure, $\varphi_i \geq 0$ (respectively, $\alpha_i \geq 0$), $i = 1, \ldots, n$, are functions (respectively, numbers such that $\sum_{i=1}^n \alpha_i \leq 1$). Then (4.11) yields

$$\begin{aligned} n_{\ell_0}(\Lambda|\xi) &\leq \sum_{\ell \in \Lambda} n_\ell(\Lambda|\xi) w_\alpha(\ell_0, \ell) \\ &\leq \frac{1}{1 - \vartheta \hat{J}_\alpha/\varkappa} \left[C_{4.1} \sum_{\ell' \in \Lambda} w_\alpha(\ell_0, \ell') + \vartheta \hat{J}_\alpha \sum_{\ell' \in \Lambda^c} |\xi_{\ell'}|^2_{L^2_\beta} w_\alpha(\ell_0, \ell') \right]. \end{aligned} \tag{4.12}$$

Therefrom, for all $\xi \in \Omega^{\mathrm{t}}$, we get

$$\begin{aligned} \limsup_{\Lambda \nearrow \mathbb{L}} n_{\ell_0}(\Lambda|\xi) &\leq \limsup_{\Lambda \nearrow \mathbb{L}} \sum_{\ell \in \Lambda} n_\ell(\Lambda|\xi) w_\alpha(\ell_0, \ell) \\ &\leq \frac{C_{4.1}}{1 - \vartheta \hat{J}_\alpha/\varkappa} \sum_\ell w_\alpha(\ell_0, \ell) \stackrel{\mathrm{def}}{=} C_{4.7}(\alpha), \end{aligned} \tag{4.13}$$

which gives (4.7) and (4.8). The proof of (4.9) is straightforward. □

Recall that the norm $\|\cdot\|_\alpha$ was defined by (2.46). Given $\alpha \in \mathcal{I}$ and $\sigma \in (0, 1/2)$, we set, c.f., Remark 2.7.,

$$\|\xi\|_{\alpha,\sigma} = \left[\sum_\ell |\xi_\ell|^2_{C^\sigma_\beta} w_\alpha(\ell_0, \ell) \right]^{1/2}. \tag{4.14}$$

Lemma 4.4. *Let the assumptions of Lemma 4.1 be satisfied. Then for every $\alpha \in \mathcal{I}$ and $\xi \in \Omega^{\mathrm{t}}$, one finds a positive $C_{4.15}(\xi)$ such that for all $\Lambda \Subset \mathbb{L}$,*

$$\int_{\Omega} \|\omega\|_{\alpha}^{2} \pi_{\Lambda}(\mathrm{d}\omega|\xi) \leq C_{4.15}(\xi). \tag{4.15}$$

Furthermore, for every $\alpha \in \mathcal{I}$, $\sigma \in (0, 1/2)$, and $\xi \in \Omega^{\mathrm{t}}$ for which the norm (4.14) is finite, one finds a $C_{4.16}(\xi) > 0$ such that for all $\Lambda \Subset \mathbb{L}$,

$$\int_{\Omega} \|\omega\|_{\alpha,\sigma}^{2} \pi_{\Lambda}(\mathrm{d}\omega|\xi) \leq C_{4.16}(\xi). \tag{4.16}$$

Proof: For any fixed $\xi \in \Omega^{\mathrm{t}}$, by the Jensen inequality and (4.12) one has

$$\begin{aligned} &\limsup_{\Lambda \nearrow \mathbb{L}} \int_{\Omega} \|\omega\|_{\alpha}^{2} \pi_{\Lambda}(\mathrm{d}\omega|\xi) \\ &\quad \leq \limsup_{\Lambda \nearrow \mathbb{L}} \left[\frac{1}{\varkappa} \sum_{\ell \in \Lambda} n_{\ell}(\Lambda|\xi) w_{\alpha}(0,\ell) + \sum_{\ell \in \Lambda^{c}} |\xi_{\ell}|_{L_{\beta}^{2}}^{2} w_{\alpha}(0,\ell) \right] \\ &\quad \leq C_{4.7}(\alpha)/\varkappa. \end{aligned} \tag{4.17}$$

Hence, the set consisting of the left-hand sides of (4.15) indexed by $\Lambda \Subset \mathbb{L}$ is bounded. The proof of (4.16) is analogous. □

4.2. Weak Convergence of Tempered Measures

Recall that $f : \Omega \to \mathbb{R}$ is a local function if it is measurable with respect to $\mathcal{B}(\Omega_{\Lambda})$ for a certain $\Lambda \Subset \mathbb{L}$.

Lemma 4.5. *Let a sequence $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{P}(\Omega^{\mathrm{t}})$ have the following properties: (a) for every $\alpha \in \mathcal{I}$, each its element obeys the estimate*

$$\int_{\Omega^{\mathrm{t}}} \|\omega\|_{\alpha}^{2} \mu_{n}(\mathrm{d}\omega) \leq C_{4.18}(\alpha), \tag{4.18}$$

with one and the same $C_{4.18}(\alpha)$; (b) for every local $f \in C_{\mathrm{b}}(\Omega^{\mathrm{t}})$, $\{\mu_n(f)\}_{n \in \mathbb{N}} \subset \mathbb{R}$ is a Cauchy sequence. Then $\{\mu_n\}_{n \in \mathbb{N}}$ converges in $\mathcal{W}^{\mathrm{t}}$ to a certain $\mu \in \mathcal{P}(\Omega^{\mathrm{t}})$.

Proof: The topology of the Polish space Ω^{t} is consistent with the following metric (c.f., (2.47))

$$\rho(\omega, \tilde{\omega}) = \sum_{k=1}^{\infty} 2^{-k} \frac{\|\omega - \tilde{\omega}\|_{\alpha_k}}{1 + \|\omega - \tilde{\omega}\|_{\alpha_k}} + \sum_{\ell} 2^{-|\ell_0 - \ell|} \frac{|\omega_{\ell} - \tilde{\omega}_{\ell}|_{C_{\beta}}}{1 + |\omega_{\ell} - \tilde{\omega}_{\ell}|_{C_{\beta}}}, \tag{4.19}$$

where $\{\alpha_k\}_{k\in\mathbb{N}} \subset \mathcal{I} = (\underline{\alpha}, \overline{\alpha})$ is a strictly decreasing sequence converging to $\underline{\alpha}$. Let us denote by $C_{\rm b}^{\rm u}(\Omega^{\rm t};\rho)$ the set of all bounded functions $f : \Omega^{\rm t} \to \mathbb{R}$ which are uniformly continuous with respect to (4.19). Thus, in accord with a known fact, see e.g. Theorem 2.1.1, page 19 of Ref. 22, to prove the lemma it suffices to show that under its conditions $\{\mu_n(f)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for every $f \in C_{\rm b}^{\rm u}(\Omega^{\rm t};\rho)$. Given $\delta > 0$, we choose $\Lambda_\delta \Subset \mathbb{L}$ and $k_\delta \in \mathbb{N}$ such that

$$\sum_{\ell\in\Lambda_\delta^c} 2^{-|\ell_0-\ell|} < \delta/3, \qquad \sum_{k=k_\delta}^{\infty} 2^{-k} = 2^{-k_\delta+1} < \delta/3. \tag{4.20}$$

For this δ and a certain $R > 0$, we choose $\Lambda_\delta(R) \Subset \mathbb{L}$ such that

$$\sup_{\ell\in\mathbb{L}\setminus\Lambda_\delta(R)} \left\{ w_{\alpha_{k_\delta-1}}(\ell_0,\ell)/w_{\alpha_{k_\delta}}(\ell_0,\ell) \right\} < \frac{\delta}{3R^2}, \tag{4.21}$$

which is possible in view of (2.37). Finally, for $R > 0$, we set

$$B_R = \{\omega \in \Omega^{\rm t} \mid \|\omega\|_{\alpha_{k_\delta}} \le R\}. \tag{4.22}$$

By (4.18) and the Chebyshev inequality, one has that for all $n \in \mathbb{N}$,

$$\mu_n(\Omega^{\rm t} \setminus B_R) \le C_{4.18}(\alpha_{k_\delta})/R^2. \tag{4.23}$$

Now for $f \in C_{\rm b}^{\rm u}(\Omega^{\rm t};\rho)$, $\Lambda \Subset \mathbb{L}$, and $n, m \in \mathbb{N}$, we have

$$\begin{aligned}|\mu_n(f) - \mu_m(f)| \le\ & |\mu_n(f_\Lambda) - \mu_m(f_\Lambda)| \\ & + 2\max\{\mu_n(|f - f_\Lambda|); \mu_m(|f - f_\Lambda|)\},\end{aligned} \tag{4.24}$$

where $f_\Lambda(\omega) \stackrel{\rm def}{=} f(\omega_\Lambda \times 0_{\Lambda^c})$. By (4.23),

$$\begin{aligned}\mu_n(|f - f_\Lambda|) \le\ & 2C_{4.18}(\alpha_{k_\delta})\|f\|_\infty/R^2 \\ & + \int_{B_R} |f(\omega) - f(\omega_\Lambda \times 0_{\Lambda^c})|\, \mu_n({\rm d}\omega).\end{aligned} \tag{4.25}$$

For chosen $f \in C_{\rm b}^{\rm u}(\Omega^{\rm t};\rho)$ and $\varepsilon > 0$, one finds $\delta > 0$ such that for all $\omega, \tilde{\omega} \in \Omega^{\rm t}$,

$$|f(\omega) - f(\tilde{\omega})| < \varepsilon/6, \quad \text{whenever} \quad \rho(\omega, \tilde{\omega}) < \delta.$$

For these f, ε, and δ, one picks up $R(\varepsilon,\delta) > 0$ such that

$$C_{4.18}(\alpha_{k_\delta})\|f\|_\infty/\,[R(\varepsilon,\delta)]^2 < \varepsilon/12. \tag{4.26}$$

Now one takes $\Lambda \Subset \mathbb{L}$, which contains both Λ_δ and $\Lambda_\delta[R(\varepsilon, \delta)]$ defined by (4.20), (4.21). For this Λ, $\omega \in B_{R(\varepsilon,\delta)}$, and $k = 1, 2, \ldots, k_\delta - 1$, one has

$$\begin{aligned} \|\omega - \omega_\Lambda \times 0_{\Lambda^c}\|^2_{\alpha_k} &= \sum_{\ell \in \Lambda^c} |\omega_\ell|^2_{L^2_\beta} w_{\alpha_{k_\delta}}(\ell_0, \ell) \left[w_{\alpha_k}(\ell_0, \ell)/w_{\alpha_{k_\delta}}(\ell_0, \ell) \right] \\ &\leq \frac{\delta}{3\,[R(\varepsilon, \delta)]^2} \sum_{\ell \in \Lambda^c} |\omega_\ell|^2_{L^2_\beta} w_{\alpha_{k_\delta}}(\ell_0, \ell) < \frac{\delta}{3}, \end{aligned} \tag{4.27}$$

where (4.21), (4.22) have been used. Then by (4.19), (4.20), it follows that

$$\forall \omega \in B_{R(\varepsilon,\delta)} : \quad \rho(\omega, \omega_\Lambda \times 0_{\Lambda^c}) < \delta, \tag{4.28}$$

which together with (4.26) yields in (4.25)

$$\mu_n(|f - f_\Lambda|) < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} \mu_n \left(B_{R(\varepsilon,\delta)} \right) \leq \frac{\varepsilon}{3}.$$

By assumption (b) of the lemma, one finds N_ε such that for all $n, m > N_\varepsilon$,

$$|\mu_n(f_\Lambda) - \mu_m(f_\Lambda)| < \frac{\varepsilon}{3}.$$

Applying the latter two estimates in (4.24) we get that $\{\mu_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in the topology $\mathcal{W}^{\mathrm{t}}$ in which $\mathcal{P}(\Omega^{\mathrm{t}})$ is complete. □

5. PROOF OF THEOREMS 3.1–3.4

The existence of Euclidean Gibbs measures and the estimate (3.1) can be proven independently. To establish the compactness of $\mathcal{G}^{\mathrm{t}}$ we will need (3.1), thus, we first prove Theorem 3.2.

Proof of Theorem 3.2: Let us show that every $\mu \in \mathcal{P}(\Omega)$ which solves the DLR equation (2.62) ought to obey (3.1) with one and the same $C_{3.1}$. To this end we apply the bounds for the kernels $\pi_\Lambda(\cdot|\xi)$ obtained above. Consider the functions

$$G_N(\omega_\ell) \stackrel{\mathrm{def}}{=} \exp\left(\min\left\{ \lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \varkappa |\omega_\ell|^2_{L^2_\beta}; N \right\} \right), \quad N \in \mathbb{N}.$$

By (2.62), Fatou's lemma, and the estimate (4.8) with an arbitrarily chosen $\alpha \in \mathcal{I}$, we get

$$\begin{aligned}
\int_{\Omega} G_N(\omega_\ell)\mu(\mathrm{d}\omega) &= \limsup_{\Lambda \nearrow \mathbb{L}} \int_{\Omega} \left[\int_{\Omega} G_N(\omega_\ell)\pi_\Lambda(\mathrm{d}\omega|\xi) \right] \mu(\mathrm{d}\xi) \\
&\leq \limsup_{\Lambda \nearrow \mathbb{L}} \int_{\Omega} \left[\int_{\Omega} \exp\left(\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \varkappa |\omega_\ell|^2_{L^2_\beta} \right) \pi_\Lambda(\mathrm{d}\omega|\xi) \right] \mu(\mathrm{d}\xi) \\
&\leq \int_{\Omega} \left[\limsup_{\Lambda \nearrow \mathbb{L}} \int_{\Omega} \exp\left(\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \varkappa |\omega_\ell|^2_{L^2_\beta} \right) \pi_\Lambda(\mathrm{d}\omega|\xi) \right] \mu(\mathrm{d}\xi) \\
&\leq \exp C_{4.7}(\alpha) \stackrel{\mathrm{def}}{=} C_{3.1}.
\end{aligned}$$

In view of the support property (2.64) of any measure solving the equation (2.62) we can pass here to the limit $N \to +\infty$ and get (3.1). $\square$

Corollary 5.1. *For every $\alpha \in \mathcal{I}$, the topologies induced on $\mathcal{G}^{\mathrm{t}}$ by $\mathcal{W}_\alpha$ and $\mathcal{W}^{\mathrm{t}}$ coincide.*

Proof: Follows immediately from Lemma 4.5 and the estimate (3.1). $\square$

Proof of Theorem 3.1: Let us introduce the next scale of Banach spaces (c.f., (2.46))

$$\Omega_{\alpha,\sigma} = \left\{ \omega \in \Omega \;\middle|\; \|\omega\|_{\alpha,\sigma} < \infty \right\}, \quad \sigma \in (0, 1/2), \;\; \alpha \in \mathcal{I}, \tag{5.1}$$

where the norm $\|\cdot\|_{\alpha,\sigma}$ was defined by (4.14). For any pair $\alpha, \alpha' \in \mathcal{I}$ such that $\alpha < \alpha'$, the embedding $\Omega_{\alpha,\sigma} \hookrightarrow \Omega_{\alpha'}$ is compact, see Remark 2.6. This fact and the estimate (4.16), which holds for any $\xi \in \Omega_{\alpha,\sigma}$, imply by Prokhorov's criterion the relative compactness of the set $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda \Subset \mathbb{L}}$ in $\mathcal{W}_{\alpha'}$. Therefore, the sequence $\{\pi_\Lambda(\cdot|0)\}_{\Lambda \Subset \mathbb{L}}$ is relatively compact in every $\mathcal{W}_\alpha$, $\alpha \in \mathcal{I}$. Then Lemma 2.13 yields $\mathcal{G}^{\mathrm{t}} \neq \emptyset$. By the same Prokhorov criterion and the estimate (3.1), we get the $\mathcal{W}_\alpha$-relative compactness of $\mathcal{G}^{\mathrm{t}}$. Then in view of the Feller property (Lemma 2.10), the set $\mathcal{G}^{\mathrm{t}}$ is closed and hence compact in every $\mathcal{W}_\alpha$, $\alpha \in \mathcal{I}$, which by Corollary 5.1 completes the proof. $\square$

Proof of Theorem 3.3: To some extent we shall follow the line of arguments used in the proof of Lemma 3.1 in Ref. 62. Given $\ell, \ell_0, b > 0$, $\sigma \in (0, 1/2)$, and $\Lambda \subset \mathbb{L}$, we introduce

$$\begin{aligned}
\Xi_\ell(\ell_0, b, \sigma) &= \left\{ \xi \in \Omega \mid |\xi_\ell|^2_{C^\sigma_\beta} \leq b \log(1 + |\ell - \ell_0|) \right\}, \\
\Xi_\Lambda(\ell_0, b, \sigma) &= \bigcap_{\ell \in \Lambda} \Xi_\ell(\ell_0, b, \sigma).
\end{aligned} \tag{5.2}$$

For a cofinal sequence $\mathcal{L}$, we set

$$\Xi(\ell_0, b, \sigma) = \bigcup_{\Lambda \in \mathcal{L}} \Xi_{\Lambda^c}(\ell_0, b, \sigma), \quad \Xi(b, \sigma) = \bigcap_{\ell_0 \in \mathbb{L}} \Xi(\ell_0, b, \sigma). \tag{5.3}$$

The latter $\Xi(b, \sigma)$ is a subset of Ω^{t} and is the same as the one given by (3.2). To prove the theorem let us show that for any $\sigma \in (0, 1/2)$, there exists $b > 0$ such that for all ℓ_0 and $\mu \in \mathcal{G}^{\mathrm{t}}$,

$$\mu\left(\Omega \setminus \Xi(\ell_0, b, \sigma)\right) = 0. \tag{5.4}$$

By (5.2) we have

$$\begin{aligned}\Omega \setminus \Xi_{\Lambda^c}(\ell_0, b, \sigma) &= \left\{\xi \in \Omega \mid (\exists \ell \in \Lambda^c) : |\xi_\ell|^2_{C^\sigma_\beta} > b \log(1 + |\ell - \ell_0|)\right\} \\ &\subset \left\{\xi \in \Omega \mid (\exists \ell \in \Delta^c) : |\xi_\ell|^2_{C^\sigma_\beta} > b \log(1 + |\ell - \ell_0|)\right\},\end{aligned} \tag{5.5}$$

for any $\Delta \subset \Lambda$. Therefore,

$$\mu\left(\bigcap_{\Lambda \in \mathcal{L}} [\Omega \setminus \Xi_{\Lambda^c}(\ell_0, b, \sigma)]\right) = \lim_{\mathcal{L}} \mu\left(\Omega \setminus \Xi_{\Lambda^c}(\ell_0, b, \sigma)\right), \tag{5.6}$$

which holds for any cofinal sequence $\mathcal{L}$. By (5.5),

$$\begin{aligned}\mu\left(\Omega \setminus \Xi_{\Lambda^c}(\ell_0, b, \sigma)\right) &= \mu\left(\bigcup_{\ell \in \Lambda^c} [\Omega \setminus \Xi_\ell(\ell_0, b, \sigma)]\right) \\ &\leq \sum_{\ell \in \Lambda^c} \mu\left(\left\{\xi \mid \exp\left(\lambda_\sigma |\xi_\ell|^2_{C^\sigma_\beta}\right) > (1 + |\ell - \ell_0|)^{b\lambda_\sigma}\right\}\right).\end{aligned}$$

Applying here the Chebyshev inequality and the estimate (3.1) we get

$$\mu\left(\Omega \setminus \Xi_{\Lambda^c}(\ell_0, b, \sigma)\right) \leq C_{3.1} \sum_{\ell \in \Lambda^c} (1 + |\ell - \ell_0|)^{-b\lambda_\sigma}.$$

In view of (2.1) the latter series converges for any $b > d/\lambda_\sigma$. In this case by (5.6)

$$\mu\left(\Omega \setminus \Xi(\ell_0, b, \sigma)\right) = \lim_{\mathcal{L}} \mu\left([\Omega \setminus \Xi_{\Lambda^c}(\ell_0, b, \sigma)]\right) = 0,$$

which yields (5.4). □

Let $\mathcal{E}$ be the set of all continuous local functions $f : \Omega^{\mathrm{t}} \to \mathbb{R}$, for which there exist $\sigma \in (0, 1/2)$, $\Delta_f \Subset \mathbb{L}$, and $D_f > 0$, such that

$$|f(\omega)|^2 \leq D_f \sum_{\ell \in \Delta_f} \exp\left(\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta}\right), \quad \text{for all} \quad \omega \in \Omega^{\mathrm{t}}, \tag{5.7}$$

where λ_σ is the same as in (2.26) and (3.1). Let also $\mathrm{ex}(\mathcal{G}^{\mathrm{t}})$ stand for the set of all extreme elements of $\mathcal{G}^{\mathrm{t}}$.

Lemma 5.2. *For every $\mu \in \mathrm{ex}(\mathcal{G}^{\mathrm{t}})$ and any cofinal sequence $\mathcal{L}$, it follows that: (a) the sequence $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda\in\mathcal{L}}$ converges in $\mathcal{W}^{\mathrm{t}}$ to this μ for μ-almost all $\xi \in \Omega^{\mathrm{t}}$; (b) for every $f \in \mathcal{E}$, one has $\lim_{\mathcal{L}} \pi_\Lambda(f|\xi) = \mu(f)$ for μ-almost all $\xi \in \Omega^{\mathrm{t}}$.*

Proof: Claim (c) of Theorem 7.12, page 122 in Ref. 36, implies that for any local $f \in C_{\mathrm{b}}(\Omega^{\mathrm{t}})$,

$$\lim_{\mathcal{L}} \pi_\Lambda(f|\xi) = \mu(f), \quad \text{for} \quad \mu-\text{almost all } \xi \in \Omega^{\mathrm{t}}. \tag{5.8}$$

Then the convergence stated in our claim (a) follows from Lemmas 4.4 and 4.5. Given $f \in \mathcal{E}$ and $N \in \mathbb{N}$, we set $\Omega_N = \{\omega \in \Omega \mid |f(\omega)| > N\}$ and

$$f_N(\omega) = \begin{cases} f(\omega) & \text{if } |f(\omega)| \le N; \\ Nf(\omega)/|f(\omega)| & \text{otherwise.} \end{cases}$$

Each f_N belongs to $C_{\mathrm{b}}(\Omega^{\mathrm{t}})$ and $f_N \to f$ point-wise as $N \to +\infty$. Then by (5.8) there exists a Borel set $\Xi_\mu \subset \Omega^{\mathrm{t}}$, such that $\mu(\Xi_\mu) = 1$ and for every $N \in \mathbb{N}$,

$$\lim_{\mathcal{L}} \pi_\Lambda(f_N|\xi) = \mu(f_N), \quad \text{for all } \xi \in \Xi_\mu. \tag{5.9}$$

Note that by (4.6), (4.9), and (5.7), for any $\xi \in \Xi_\mu$ one finds a positive $C_{5.10}(f, \xi)$ such that for all $\Lambda \Subset \mathbb{L}$, which contain Δ_f, it follows that

$$\int_\Omega |f(\omega)|^2 \pi_\Lambda(\mathrm{d}\omega|\xi) \le C_{5.10}(f, \xi). \tag{5.10}$$

Hence

$$\begin{aligned} |\pi_\Lambda(f|\xi) - \pi_\Lambda(f_N|\xi)| &\le 2 \int_{\Omega_N} |f(\omega)| \pi_\Lambda(\mathrm{d}\omega|\xi) \\ &\le \frac{2}{N} \cdot \int_\Omega |f(\omega)|^2 \pi_\Lambda(\mathrm{d}\omega|\xi) \le \frac{2}{N} \cdot C_{5.10}(f, \xi). \end{aligned}$$

Similarly, by means of (5.7) and Theorem 3.2, one gets

$$|\mu(f) - \mu(f_N)| \le \frac{2}{N} \cdot D_f C_{3.1}.$$

The latter two inequalities and (5.9) allow us to estimate $|\pi_\Lambda(f|\xi) - \mu(f)|$ and thereby to complete the proof. $\square$

Proof of Theorem 3.4: For the scalar translation invariant version of the model considered here, the high-temperature uniqueness was proven in Refs. 11, 12 by means of Dobrushin's criterium. The proof given below is a modification of the arguments used there.

The main idea of the method of Dobrushin is to control the Wasserstein distance $R[\pi_\ell(\cdot|\xi); \pi_\ell(\cdot|\xi')]$ between the measures $\pi_\ell(\cdot|\xi)$ and $\pi_\ell(\cdot|\xi')$ with $\xi \neq \xi'$.

In our context, its appropriate choice may be made as follows. For given ℓ and $\xi, \xi' \in \Omega^{\mathrm{t}}$, we set

$$R[\pi_\ell(\cdot|\xi); \pi_\ell(\cdot|\xi')] = \sup_{f \in \mathrm{Lip}_1(L^2_\beta)} \left| \int_\Omega f(\omega_\ell)\pi_\ell(\mathrm{d}\omega|\xi) - \int_\Omega f(\omega_\ell)\pi_\ell(\mathrm{d}\omega|\xi') \right|, \tag{5.11}$$

where $\mathrm{Lip}_1(L^2_\beta)$ stands for the set of Lipschitz-continuous functions $f : L^2_\beta \to \mathbb{R}$ with the Lipschitz constant equal one. The Dobrushin criterion (see Theorem 4 in Ref. 25) employs the matrix

$$C_{\ell\ell'} = \sup \left\{ \frac{R[\pi_\ell(\cdot|\xi); \pi_\ell(\cdot|\xi')]}{|\xi_\ell - \xi_{\ell'}|_{L^2_\beta}} \right\}, \quad \ell \neq \ell', \quad \ell, \ell' \in \mathbb{L}, \tag{5.12}$$

where the supremum is taken over all $\xi, \xi' \in \Omega^{\mathrm{t}}$ which differ only at ℓ'. According to this criterium the uniqueness stated will follow if

$$\sup_\ell \sum_{\ell' \in \mathbb{L} \setminus \{\ell\}} C_{\ell\ell'} < 1. \tag{5.13}$$

In view of (2.57) the map

$$L^2_\beta \ni \xi_{\ell'} \mapsto \Upsilon(\xi_{\ell'}) \stackrel{\mathrm{def}}{=} \int_\Omega f(\omega_\ell)\pi_\ell(\mathrm{d}\omega|\xi) \tag{5.14}$$

has the following derivative in direction $\zeta \in L^2_\beta$

$$(\nabla\Upsilon(\xi_{\ell'}), \zeta)_{L^2_\beta} = -J_{\ell\ell'}\left[\pi_\ell\left(f \cdot (\omega_\ell, \zeta)_{L^2_\beta}|\xi\right) - \pi_\ell(f|\xi) \cdot \pi_\ell\left((\omega_\ell, \zeta)_{L^2_\beta}|\xi\right)\right].$$

By Theorem 5.1 of Ref. 11, the measures $\pi_\ell(\cdot|\xi)$ obey the logarithmic Sobolev inequality with the constant

$$C_{\mathrm{LS}} = e^{\beta\delta}/(a+b), \tag{5.15}$$

which is independent of ξ. By standard arguments this yields the estimate

$$|(\nabla\Upsilon(\xi_{\ell'}), \zeta)_{L^2_\beta}| \leq C_{\mathrm{LS}}|J_{\ell\ell'}| \cdot |\zeta|^2_{L^2_\beta}. \tag{5.16}$$

Then with the help of the mean value theorem from (5.12) and (5.15) we get

$$C_{\ell\ell'} \leq |J_{\ell\ell'}| \cdot e^{\beta\delta}/(a+b).$$

Thereby, the validity of the uniqueness condition (5.13) is ensured by (3.7). $\square$

6. PROOF OF THEOREMS 3.8–3.10

6.1. Stochastic Order and the Proof of Theorem 3.8

First we prove that the cone $K_+(\Omega^{\mathrm{t}})$ may be used to establish an order on $\mathcal{G}^{\mathrm{t}}$, that is it has the property: if $\mu(f) \leq \tilde{\mu}(f)$ and $\tilde{\mu}(f) \leq \mu(f)$ for all $f \in K_+(\Omega^{\mathrm{t}})$, then $\mu = \tilde{\mu}$.

Proof of Lemma 3.6: Let us show that the cone $K_+(\Omega^{\mathrm{t}})$ contains a defining class for $\mathcal{G}^{\mathrm{t}}$. Usually, measure defining classes of functions are established by means of monotone class theorems, see e.g., Ref. 19, pages 36–39. In our situation, a sufficient condition for a set of bounded continuous functions to be a measure defining class may be formulated as follows: is should (a) contain constant functions; (b) be closed under multiplication; (c) separate points of Ω^{t}. The class (3.10) does not meet (b); hence, to prove the stated one has to use additional arguments.

A continuous function $f : \Omega^{\mathrm{t}} \to \mathbb{R}$ is called a cylinder function if it possesses the representation

$$f(\omega) = \phi(\omega_{\ell_1}(\tau_1), \ldots, \omega_{\ell_n}(\tau_n)), \tag{6.1}$$

with certain $n \in \mathbb{N}$, $\ell_1, \ldots, \ell_n$, $\tau_1, \ldots, \tau_n$, and a continuous $\phi : \mathbb{R}^n \to \mathbb{R}$. By $K_+^{\mathrm{cyl}}(\Omega^{\mathrm{t}})$ we denote the subset of $K_+(\Omega^{\mathrm{t}})$ consisting of cylinder functions. Suppose that the equality (3.11) holds for all $f \in K_+^{\mathrm{cyl}}(\Omega^{\mathrm{t}})$. Then

$$\int_{\Omega^{\mathrm{t}}} \omega_\ell(\tau)\mu(\mathrm{d}\omega) = \int_{\Omega^{\mathrm{t}}} \omega_\ell(\tau)\tilde{\mu}(\mathrm{d}\omega), \quad \text{for all} \quad \ell, \tau, j. \tag{6.2}$$

For fixed $\ell_1, \ldots, \ell_n$ and $\tau_1, \ldots, \tau_n$, let P and $\tilde{P}$ be the projections of the measures μ and $\tilde{\mu}$ on $\mathbb{R}^n$. That is, each of P and $\tilde{P}$ obeys

$$\int_{\Omega^{\mathrm{t}}} f(\omega)\mu(\mathrm{d}\omega) = \int_{\mathbb{R}^n} \phi(x_1, \ldots, x_n)P(\mathrm{d}x),$$

for f and ϕ as in (6.1). Then by (3.11), it follows that

$$\int_{\mathbb{R}^n} \phi(x_1, \ldots, x_n)P(\mathrm{d}x) \leq \int_{\mathbb{R}^n} \phi(x_1, \ldots, x_n)\tilde{P}(\mathrm{d}x), \tag{6.3}$$

for all increasing ϕ. Let $\widehat{P}$ be a probability measure on $\mathbb{R}^{2n}$, such that

$$P(\mathrm{d}x) = \int_{\mathbb{R}^n} \widehat{P}(\mathrm{d}x, \mathrm{d}\tilde{x}), \quad \tilde{P}(\mathrm{d}\tilde{x}) = \int_{\mathbb{R}^n} \widehat{P}(\mathrm{d}x, \mathrm{d}\tilde{x}).$$

Thus, $\widehat{P}$ is a *coupling* of P and $\tilde{P}$. Of course, the above equalities do not determine $\widehat{P}$ uniquely. By the Kantorovich-Rubinstein duality theorem, the Wasserstein distance, c.f., (5.11), between the measures P and $\tilde{P}$ which have first moments, can be defined as follows, see Ref. 28,

$$R(P, \tilde{P}) = \inf \int_{\mathbb{R}^{2n}} |x - \tilde{x}|\widehat{P}(\mathrm{d}x, \mathrm{d}\tilde{x}), \tag{6.4}$$

where infimum is taken over all couplings of P and $\tilde{P}$. It is a metric, and the convergence of a sequence of measures in this metric is equivalent to its weak convergence combined with the convergence of the first moments. Consider

$$M = \{(x, \tilde{x}) \in \mathbb{R}^{2n} \mid x_i \leq \tilde{x}_i, \quad \text{for all} \quad i = 1, \ldots, n\}.$$

As this set is closed in $\mathbb{R}^{2n}$, by Strassen's theorem (see page 129 of Ref. 64), from (6.3) it follows that there exists a coupling $\widehat{P}_*$ such that

$$\widehat{P}_*(M) = 1. \tag{6.5}$$

Thereby,

$$\begin{aligned} R(P, \tilde{P}) &\leq \int_M |x - \tilde{x}| \widehat{P}_*(\mathrm{d}x, \mathrm{d}\tilde{x}) \\ &\leq \sum_{i=1}^n \int_{\mathbb{R}^{2n}} (\tilde{x}_i - x_i) \widehat{P}_*(\mathrm{d}x, \mathrm{d}\tilde{x}) \\ &= \sum_{i=1}^n \int_{\mathbb{R}^n} x_i \left[\tilde{P}(\mathrm{d}x) - P(\mathrm{d}x)\right] = 0. \end{aligned}$$

The latter equality follows from (6.2). Since the subset of $C_{\mathrm{b}}(\Omega^{\mathrm{t}})$ consisting of all cylinder functions (6.1) is a defining class for $\mathcal{P}(\Omega^{\mathrm{t}})$, this yields $\mu = \tilde{\mu}$. $\square$

One observes that for (6.3) to hold, it was enough to have $\mu \leq \tilde{\mu}$, c.f., (3.10). Thus, we have one more important fact arising from the proof of the above lemma.

Corollary 6.1. *If for any $\mu, \tilde{\mu} \in \mathcal{G}^{\mathrm{t}}$, such that $\mu \leq \tilde{\mu}$, all their first moments coincide, i.e., (6.2) holds, then $\mu = \tilde{\mu}$.*

Remark 6.2. *For every ℓ, $t_\ell(\omega) \leq t_\ell(\tilde{\omega})$ if $\omega \leq \tilde{\omega}$. This means that the transformation θ_ℓ defined in (2.65) is order preserving.*

Proof of Theorem 3.8: In establishing the existence of the elements $\mu_\pm$ the main point was to prove Lemma 3.6. Thereby, the existence of $\mu_\pm$ can be proven by literal repetition of the arguments used in Ref. 17 for proving Theorem IV.3. They are unique by definition. Indeed, for two maximal elements, say μ_+ and $\tilde{\mu}_+$, one would have $\mu_+ \leq \tilde{\mu}_+$ and $\tilde{\mu}_+ \leq \mu_+$ at the same time. Thus, $\mu_+ = \tilde{\mu}_+$. The proof of the extremeness (respectively, the symmetry properties) of $\mu_\pm$ can be done by following the proof of Proposition V.1 (respectively, Proposition V.3) in Ref. 17. Some additional properties of $\mu_\pm$ will be described in the subsequent section. $\square$

The result just proven and Corollary 6.2 yield the following

Lemma 6.3. *Suppose that, for all ℓ,*

$$\mu_+(\omega_\ell(0)) = \mu_-(\omega_\ell(0)). \tag{6.6}$$

Then $\mathcal{G}^{\mathrm{t}}$ is a singleton. If the model is symmetric, then (6.6) turns into

$$\mu_+(\omega_\ell(0)) = \mu_-(\omega_\ell(0)) = 0.$$

6.2. Existence of Pressure and the Proof of Theorem 3.10

Here we consider a translation invariant version of our model. Given $R > 0$ and $\Lambda \Subset \mathbb{L}$, let $\partial_R^+ \Lambda$ be the set of all $\ell \in \Lambda^c$, such that $\mathrm{dist}(\ell, \Lambda) \leq R$. Then for a van Hove sequence $\mathcal{L}$ and any $R > 0$, one has $\lim_{\mathcal{L}} |\partial_R^+ \Lambda|/|\Lambda| = 0$, yielding

$$\lim_{\mathcal{L}} \frac{1}{|\Lambda|} \sum_{\ell \in \Lambda, \ell' \in \Lambda^c} J_{\ell\ell'} = 0. \tag{6.7}$$

The existence of van Hove sequences means the amenability of the graph $(\mathbb{L}, E)$, E being the set of all pairs ℓ, ℓ', such that $|\ell - \ell'| = 1$. For nonamenable graphs, phase transitions with $h \neq 0$ are possible; hence, statements like Theorem 3.14 do not hold, see Refs. 43, 65.

Let us prove first the existence of the pressure corresponding to the zero boundary conditions.

Lemma 6.4. *For every $h \in \mathbb{R}$, the limiting pressure $p(h) = \lim_{\mathcal{L}} p_\Lambda(h)$ exists for every van Hove sequence $\mathcal{L}$. It is independent of the particular choice of $\mathcal{L}$.*

Proof: For $t \geq 0$, $\xi \in \Omega^{\mathrm{t}}$, and $\Delta \subset \Lambda$, let $\varpi^{(t)}_{\Lambda,\Delta}$, $Y_{\Lambda,\Delta}(t)$ be defined by (7.24) below with the potentials $V_\ell = V$ having the form (3.36). Then we set

$$f_{\Lambda,\Delta}(t) = \frac{1}{|\Lambda|} \log Y_{\Lambda,\Delta}(t), \quad t \geq 0. \tag{6.8}$$

This function is differentiable and

$$g_{\Lambda,\Delta}(t) \stackrel{\text{def}}{=} f'_{\Lambda,\Delta}(t) = \frac{1}{2|\Lambda|} \sum_{\ell,\ell' \in \Delta} J_{\ell\ell'} \varpi^{(t)}_{\Lambda,\Delta}\big[(\omega_\ell, \omega_{\ell'})_{L^2_\beta}\big]$$
$$+ \frac{1}{|\Lambda|} \sum_{\ell \in \Delta, \ell' \in \Lambda \setminus \Delta} J_{\ell\ell'} \varpi^{(t)}_{\Lambda,\Delta}\big[(\omega_\ell, \omega_{\ell'})_{L^2_\beta}\big] \geq 0. \tag{6.9}$$

Here we used that $\varpi^{(t)}_{\Lambda,\Delta}[(\omega_\ell, \omega_{\ell'})_{L^2_\beta}] \geq 0$, which follows from the GKS inequality (7.4). The function $g_{\Lambda,\Delta}$ is also differentiable and

$$g'_{\Lambda,\Delta}(t) \geq 0, \tag{6.10}$$

which may be proven similarly by means of the GKS inequality (7.5). Therefore,

$$f_{\Lambda,\Delta}(0) \leq f_{\Lambda,\Delta}(1) \leq g_{\Lambda,\Delta}(1). \tag{6.11}$$

Now we take here $\Delta = \Lambda$ and obtain that p_Λ is a convex function of h. Furthermore, by (4.15), for any $\alpha \in \mathcal{I}$,

$$\log Y_{\{\ell\},\{\ell\}}(0) \leq p_\Lambda(h) \leq \hat{J}_0 C_{4.15}(0)/2. \tag{6.12}$$

By the translation invariance the lower bound in (6.12) is independent of ℓ. Therefore, the set $\{p_\Lambda(h)\}_{\Lambda\Subset\mathbb{L}}$ has accumulation points. For one of them, $p(h)$, let $\{\Gamma_n\}_{n\in\mathbb{N}}$ be the sequence of parallelepipeds such that $p_{\Gamma_n}(h) \to p(h)$ as $n \to +\infty$. Let also $\mathcal{L}$ be a van Hove sequence. Given $n \in \mathbb{N}$ and $\Lambda \in \mathcal{L}$, let $\mathfrak{L}_n^-(\Lambda) \subset \mathfrak{G}(\Gamma_n)$ (respectively, $\mathfrak{L}_n^+(\Lambda) \subset \mathfrak{G}(\Gamma_n)$) consist of the translates of Γ_n which are contained in Λ (respectively, which have non-void intersections with Λ). Let also

$$\Lambda_n^\pm = \bigcup_{\Gamma\in\mathfrak{L}_n^\pm} \Gamma. \tag{6.13}$$

Now we take in (6.8) first $\Delta = \Lambda_n^-$, then $\Delta = \Lambda$, $\Lambda = \Lambda_n^+$, and obtain by (6.11)

$$\frac{|\Lambda_n^-|}{|\Lambda|} p_{\Lambda_n^-}(h) \leq p_\Lambda(h) \leq \frac{|\Lambda_n^+|}{|\Lambda|} p_{\Lambda_n^+}(h). \tag{6.14}$$

Let us estimate $p_{\Lambda_n^\pm}(h) - p_{\Gamma_n}(h)$. To this end we introduce for $t \geq 0$, c.f., (7.24),

$$\begin{aligned} X_{\Lambda_n^-}(t) = \int_{\Omega_{\Lambda_n^-}} \exp\Bigg\{ & \frac{1}{2} \sum_{\Gamma\in\mathfrak{L}_n^-} \sum_{\ell,\ell'\in\Gamma} J_{\ell\ell'}(\omega_\ell, \omega_{\ell'})_{L^2_\beta} \\ & + t \sum_{\Gamma,\Gamma'\in\mathfrak{L}_n^-,\ \Gamma\neq\Gamma'} \sum_{\ell\in\Gamma} \sum_{\ell'\in\Gamma'} J_{\ell\ell'}(\omega_\ell, \omega_{\ell'})_{L^2_\beta} \\ & + \sum_{\ell\in\Lambda_n^-} \int_0^\beta \left[h\omega_\ell(\tau) - v([\omega_\ell(\tau)]^2)\right] \mathrm{d}\tau \Bigg\} \chi_{\Lambda_n^-}(\mathrm{d}\omega), \end{aligned} \tag{6.15}$$

and

$$f_{\Lambda_n^-}(t) = \frac{1}{|\Lambda_n^-|} \log X_{\Lambda_n^-}(t). \tag{6.16}$$

Then

$$f_{\Lambda_n^-}(1) = p_{\Lambda_n^-}(h), \quad f_{\Lambda_n^-}(0) = \frac{|\Gamma_n|}{|\Lambda_n^-|} \sum_{\Gamma\in\mathfrak{L}_n^-} p_\Gamma(h) = p_{\Gamma_n}(h). \tag{6.17}$$

Observe that $p_\Gamma(h) = p_{\Gamma_n}(h)$ for all $\Gamma \in \mathfrak{G}(\Gamma_n)$, which follows from the translation invariance of the model. Thereby,

$$\begin{aligned}
0 &\leq p_{\Lambda_n^-}(h) - p_{\Gamma_n}(h) \leq f'_{\Lambda_n^-}(1) \\
&= \frac{1}{|\Lambda_n^-|} \sum_{\Gamma,\Gamma' \in \mathfrak{L}_n^-,\ \Gamma \neq \Gamma'} \sum_{\ell \in \Gamma} \sum_{\ell' \in \Gamma'} J_{\ell\ell'} \pi_{\Lambda_n^-}\big((\omega_\ell, \omega_{\ell'})_{L^2_\beta} |0\big) \\
&\leq \frac{1}{|\Lambda_n^-|} \sum_{\Gamma \in \mathfrak{L}_n^-} \sum_{\ell \in \Gamma} \sum_{\ell' \in \Gamma^c} J_{\ell\ell'} \pi_{\Lambda_n^-}\big((\omega_\ell, \omega_{\ell'})_{L^2_\beta} |0\big) \\
&\leq \hat{J}(\Gamma_n) C_{4.15}(0),
\end{aligned} \tag{6.18}$$

where we used the estimate (4.15) and set

$$\hat{J}(\Gamma_n) = \frac{1}{|\Gamma_n|} \sum_{\ell \in \Gamma_n} \sum_{\ell' \in \Gamma_n^c} J_{\ell\ell'} = \frac{1}{|\Gamma|} \sum_{\ell \in \Gamma} \sum_{\ell' \in \Gamma^c} J_{\ell\ell'}, \quad \text{for every} \quad \Gamma \in \mathfrak{G}(\Gamma_n). \tag{6.19}$$

In deriving (6.18) we took into account that the function (6.16) has positive first and second derivatives, c.f., (6.9) and (6.10). By literal repetition one proves that both estimates from (6.18) hold also for $p_{\Lambda_n^+}(h) - p_{\Gamma_n}(h)$. In view of (6.7) the above $\hat{J}(\Gamma_n)$ may be made arbitrarily small by taking big enough Γ_n. Thereby, for any $\varepsilon > 0$, one can choose $n \in \mathbb{N}$ such that the following estimates hold (recall that $p_{\Gamma_n} \to p$ as $n \to +\infty$)

$$|p_{\Gamma_n}(h) - p(h)| < \varepsilon/3, \quad 0 \leq p_{\Lambda_n^-}(h) - p_{\Gamma_n}(h) \leq p_{\Lambda_n^+}(h) - p_{\Gamma_n}(h) < \varepsilon/3. \tag{6.20}$$

As $\mathcal{L}$ is a van Hove sequence, one can pick up $\Lambda \in \mathcal{L}$ such that

$$\max\left\{ \left(\frac{|\Lambda_n^+|}{|\Lambda|} - 1 \right) p_{\Lambda_n^+}(h); \left(1 - \frac{|\Lambda_n^-|}{|\Lambda|} \right) p_{\Lambda_n^+}(h) \right\} < \varepsilon/3,$$

which is possible in view of (6.12). Then for the chosen n and $\Lambda \in \mathcal{L}$, one has

$$\begin{aligned}
|p_\Lambda(h) - p(h)| &\leq |p_{\Gamma_n}(h) - p(h)| + p_{\Lambda_n^+}(h) - p_{\Gamma_n}(h) \\
&\quad + \max\left\{ \left(\frac{|\Lambda_n^+|}{|\Lambda|} - 1 \right) p_{\Lambda_n^+}(h); \left(1 - \frac{|\Lambda_n^-|}{|\Lambda|} \right) p_{\Lambda_n^+}(h) \right\} < \varepsilon,
\end{aligned}$$

which obviously holds also for all $\Lambda' \in \mathcal{L}$ such that $\Lambda \subset \Lambda'$. □

Proof of Theorem 3.10: The proof will be done if we show that, for every $\mu \in \mathcal{G}^{\mathrm{t}}$ and any van Hove sequence $\mathcal{L}$,

$$\lim_{\mathcal{L}} p_\Lambda^\mu(h) = p(h).$$

By the Jensen inequality one obtains for $t_1, t_2 \in \mathbb{R}$, $\xi \in \Omega^{\mathrm{t}}$,

$$Z_\Lambda((t_1 + t_2)\xi) \geq Z_\Lambda(t_1\xi) \exp\left\{ t_2 \sum_{\ell \in \Lambda, \ell' \in \Lambda^c} J_{\ell\ell'} \pi_\Lambda\big[(\omega_\ell, \xi_{\ell'})_{L^2_\beta} \,|t_1\xi\ \big] \right\}.$$

We set here first $t_1 = 0,\ t_2 = 1$, then $t_1 = -t_2 = 1$, and obtain after taking logarithm and dividing by $|\Lambda|$

$$\begin{aligned} p_\Lambda(h) + \frac{1}{|\Lambda|} \sum_{\ell \in \Lambda, \ell' \in \Lambda^c} J_{\ell\ell'} \pi_\Lambda\big[(\omega_\ell, \xi_{\ell'})_{L^2_\beta} |0\big] &\leq p_\Lambda(h, \xi) \\ &\leq p_\Lambda(h) + \frac{1}{|\Lambda|} \sum_{\ell \in \Lambda, \ell' \in \Lambda^c} J_{\ell\ell'} \pi_\Lambda\big[(\omega_\ell, \omega_{\ell'})_{L^2_\beta} |\xi\big], \end{aligned} \tag{6.21}$$

where we used that $\pi_\Lambda[(\omega_\ell, \omega_{\ell'})_{L^2_\beta} |\xi] = \pi_\Lambda[(\omega_\ell, \xi_{\ell'})_{L^2_\beta} |\xi]$, see (2.54). Thereby, we integrate (6.21) with respect to $\mu \in \mathcal{G}^{\mathrm{t}}$, take into account (2.3), and obtain after some calculations the following

$$\begin{aligned} p_\Lambda(h) - \frac{1}{|\Lambda|} \sum_{\ell \in \Lambda, \ell' \in \Lambda^c} J_{\ell\ell'} \pi_\Lambda\big(|\omega_\ell|_{L^2_\beta} \,|0\big)\ \mu\big(|\xi_{\ell'}|_{L^2_\beta}\big) &\leq p^\mu_\Lambda \\ &\leq p_\Lambda(h) + \frac{1}{|\Lambda|} \sum_{\ell \in \Lambda, \ell' \in \Lambda^c} J_{\ell\ell'} \mu\big((\omega_\ell, \omega_{\ell'})_{L^2_\beta}\big). \end{aligned} \tag{6.22}$$

By means of Theorem 3.2 (respectively, Lemma 4.4), one estimates $\mu((\omega_\ell, \omega_{\ell'})_{L^2_\beta})$, $\mu(|\xi_{\ell'}|_{L^2_\beta})$ (respectively, $\pi_\Lambda(|\omega_\ell|_{L^2_\beta}|0)$) by positive constants independent of ℓ, ℓ'. Thereby, the property stated follows from (6.7) and Lemma 6.4. □

Proof of Corollary 3.11: By (3.13),

$$\frac{\partial}{\partial h} p_\Lambda(h, \xi) = \frac{1}{|\Lambda|} \sum_{\ell \in \Lambda} \int_0^\beta \pi_\Lambda(\omega_\ell(\tau)|\xi)\, \mathrm{d}\tau.$$

Then, for every $\mu \in \mathcal{G}^{\mathrm{t}}$ and $\Lambda \Subset \mathbb{L}$, one has

$$\begin{aligned} \frac{\partial}{\partial h} p^\mu_\Lambda(h) &= \int_\Omega \frac{\partial}{\partial h} \big(p^\mu_\Lambda(h, \xi)\big)\, \mu(\mathrm{d}\xi) \\ &= \frac{1}{|\Lambda|} \sum_{\ell \in \Lambda} \int_0^\beta \int_\Omega \pi_\Lambda\, [\omega_\ell(\tau)|\xi]\, \mu(\mathrm{d}\xi)\mathrm{d}\tau \\ &= \frac{1}{|\Lambda|} \sum_{\ell \in \Lambda} \int_0^\beta \mu\, [\omega_\ell(\tau)]\, \mathrm{d}\tau \end{aligned} \tag{6.23}$$

By Theorem 3.10, it follows that

$$\frac{\partial}{\partial h} p^{\mu_+}(h) = \frac{\partial}{\partial h} p^{\mu_-}(h). \tag{6.24}$$

Both extreme measures $\mu_\pm$ are translation and shift invariant. Then combining (6.24) and (6.23) one obtains $\mu_+(\omega_\ell(0)) = \mu_-(\omega_\ell(0))$ for any $h \neq 0$. By Lemma 6.3 this gives the proof. □

7. PROOF OF THEOREMS 3.12, 3.13

We prove these theorem by comparing the model considered with a certain model, for which the property desired is being proven directly. The comparison is based on correlation inequalities, which we present in the next subsections. They were proven in the framework of the lattice approximation technique, analogous to that of Euclidean quantum fields.[79]

Recall that Theorems 3.12–3.14 describe the model with $\nu = 1$ and $J_{\ell\ell'} \geq 0$, which will tacitly be assumed in the statements below.

7.1. Correlation Inequalities

We begin with the FKG inequality, Theorem 6.1 in Ref. 4. Recall that the family of functions $K_+(\Omega)$ and $K_+^{\mathrm{cyl}}(\Omega)$ were introduced in (3.10) and in the proof of Lemma 3.6.

Proposition 7.1. *For all $\Lambda \Subset \mathbb{L}$, $\xi \in \Omega^{\mathrm{t}}$ and any $f, g \in K_+(\Omega)$, it follows that*

$$\pi_\Lambda(f \cdot g|\xi) \geq \pi_\Lambda(f|\xi) \cdot \pi_\Lambda(g|\xi). \tag{7.1}$$

This inequality holds also for any continuous increasing functions, for which the corresponding integrals exist. This yields in particular that for all such functions,

$$\xi \leq \tilde{\xi} \implies \pi_\Lambda(f|\xi) \leq \pi_\Lambda(f|\tilde{\xi}). \tag{7.2}$$

Next, there follow the GKS inequalities, Theorem 6.2 in Ref. 4.

Proposition 7.2. *Let the anharmonic potentials have the form*

$$V_\ell(x) = v_\ell(x^2) - h_\ell x, \quad h_\ell \geq 0 \quad \textit{for all} \quad \ell \in \mathbb{L}, \tag{7.3}$$

with v_ℓ being continuous. Let also the continuous functions $f_1, \ldots, f_{n+m} : \mathbb{R} \to \mathbb{R}$ be polynomially bounded and such that every f_i is either an odd increasing function on $\mathbb{R}$ or an even positive function, increasing on $[0, +\infty)$. Then the following *inequalities hold for all* $\tau_1, \ldots, \tau_{n+m} \in [0, \beta]$*, and all* $\ell_1, \ldots, \ell_{n+m} \in \Lambda$*,*

$$\int_\Omega \left(\prod_{i=1}^{n} f_i(\omega_{\ell_i}(\tau_i)) \right) \pi_\Lambda\,(\mathrm{d}\omega|0) \geq 0; \tag{7.4}$$

$$\int_\Omega \left(\prod_{i=1}^{n} f_i(\omega_{\ell_i}(\tau_i)) \right) \cdot \left(\prod_{i=n+1}^{n+m} f_i(\omega_{\ell_i}(\tau_i)) \right) \pi_\Lambda\,(\mathrm{d}\omega|0)$$
$$\geq \int_\Omega \left(\prod_{i=1}^{n} f_i(\omega_{\ell_i}(\tau_i)) \right) \pi_\Lambda\,(\mathrm{d}\omega|0) \cdot \int_\Omega \left(\prod_{i=n+1}^{n+m} f_i(\omega_{\ell_i}(\tau_i)) \right) \pi_\Lambda\,(\mathrm{d}\omega|0). \tag{7.5}$$

Given $\xi \in \Omega^{\mathrm{t}}$, $\Lambda \Subset \mathbb{L}$, and $\ell, \ell', \tau, \tau' \in [0, \beta]$, the pair correlation function is

$$K^\Lambda_{\ell\ell'}(\tau, \tau'|\xi) = \int_\Omega \omega_\ell(\tau)\omega_{\ell'}(\tau')\pi_\Lambda(\mathrm{d}\omega|\xi)$$
$$- \int_\Omega \omega_\ell(\tau)\pi_\Lambda(\mathrm{d}\omega|\xi) \cdot \int_\Omega \omega_{\ell'}(\tau')\pi_\Lambda(\mathrm{d}\omega|\xi). \tag{7.6}$$

Then, by (7.2),

$$K^\Lambda_{\ell\ell'}(\tau, \tau'|\xi) \geq 0, \tag{7.7}$$

which holds for all ℓ, ℓ', τ, τ', and $\xi \in \Omega^{\mathrm{t}}$. The following result is a version of the estimate (12.129), page 254 of Ref. 31, which for the Euclidean Gibbs measures may be proven by means of the lattice approximation.

Proposition 7.3. *Let* V_ℓ *be of the form (7.3) with* $h_\ell = 0$ *and the functions* v_ℓ *being convex. Then for all* ℓ, ℓ', τ, τ' *and for any* $\xi \in \Omega^{\mathrm{t}}$ *such that* $\xi \geq 0$*, it follows that*

$$K^\Lambda_{\ell\ell'}(\tau, \tau'|\xi) \leq K^\Lambda_{\ell\ell'}(\tau, \tau'|0). \tag{7.8}$$

Let us consider

$$U^\Lambda_{\ell_1\ell_2\ell_3\ell_4}(\tau_1, \tau_2, \tau_3, \tau_4) = \int_\Omega \omega_{\ell_1}(\tau_1)\omega_{\ell_2}(\tau_2)\omega_{\ell_3}(\tau_3)\omega_{\ell_4}(\tau_4)\pi_\Lambda(\mathrm{d}\omega|0)$$
$$- K^\Lambda_{\ell_1\ell_2}(\tau_1, \tau_2|0)K^\Lambda_{\ell_3\ell_4}(\tau_3, \tau_4|0)$$
$$- K^\Lambda_{\ell_1\ell_3}(\tau_1, \tau_3|0)K^\Lambda_{\ell_2\ell_4}(\tau_2, \tau_4|0)$$
$$- K^\Lambda_{\ell_1\ell_4}(\tau_1, \tau_4|0)K^\Lambda_{\ell_2\ell_3}(\tau_2, \tau_3|0), \tag{7.9}$$

which is the Ursell function for the measure $\pi_\Lambda(\cdot|0)$. The next statement gives the Gaussian domination and Lebowitz inequalities, see Ref. 4.

Proposition 7.4. *Let V_ℓ be of the form (7.3) with $h_\ell = 0$ and the functions v_ℓ being convex. Then for all $n \in \mathbb{N}$, $\ell_1, \ldots, \ell_{2n} \in \Lambda \Subset \mathbb{L}$, $\tau_1, \ldots, \tau_{2n} \in [0, \beta]$, it follows that*

$$\int_\Omega \omega_{\ell_1}(\tau_1)\omega_{\ell_2}(\tau_2)\cdots\omega_{\ell_{2n}}(\tau_{2n})\pi_\Lambda(\mathrm{d}\omega|0)$$

$$\leq \sum_\sigma \prod_{j=1}^n \int_\Omega \omega_{\ell_{\sigma(2j-1)}}\big(\tau_{\sigma(2j-1)}\big)\omega_{\ell_{\sigma(2j)}}\big(\tau_{\sigma(2j)}\big)\pi_\Lambda(\mathrm{d}\omega|0), \tag{7.10}$$

where the sum runs through the set of all partitions of $\{1, \ldots, 2n\}$ onto unordered pairs. In particular,

$$U^\Lambda_{\ell_1\ell_2\ell_3\ell_4}(\tau_1, \tau_2, \tau_3, \tau_4) \leq 0. \tag{7.11}$$

7.2. More on Extreme Elements

Here we continue to study the properties of $\mu_\pm$, the existence of which was established in Theorem 3.8. In particular, we give an explicit construction of these measures.

For ℓ_0 and $b > 0$, let $\hat{\xi} = (\hat{\xi}_\ell)_{\ell\in\mathbb{L}}$ be the following constant (with respect to $\tau \in S_\beta$) configuration

$$\hat{\xi}_\ell(\tau) = [b\log(1 + |\ell - \ell_0|)]^{1/2}. \tag{7.12}$$

Fix $\sigma \in (0, 1/2)$ and b obeying the condition $b > d/\lambda_\sigma$ (see the proof of Theorem 3.3). In view of (2.38), $\hat{\xi}$ belongs to Ω^{t}. It also belongs to $\Xi(\ell_0, b, \sigma)$, and for all $\xi \in \Xi(b, \sigma)$, one finds $\Delta \Subset \mathbb{L}$ such that $\xi^{(j)}_\ell(\tau) \leq \hat{\xi}^{(j)}_\ell(\tau)$ for all τ, j and $\ell \in \Delta^c$. Therefore, for any cofinal sequence $\mathcal{L}$ and $\xi \in \Xi(b, \sigma)$, one finds $\Delta \in \mathcal{L}$ such that for all $\Lambda \in \mathcal{L}$, $\Delta \subset \Lambda$, one has $\pi_\Lambda(\cdot|\xi) \leq \pi_\Lambda(\cdot|\hat{\xi})$, see (7.2). As was established in the proof of Theorem 3.1, every sequence $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda\in\mathcal{L}}$, $\xi \in \Xi(b, \sigma) \subset \Omega^{\mathrm{t}}$, is relatively compact in any $\mathcal{W}_\alpha$, $\alpha \in \mathcal{I}$, which by Lemmas 4.4, 4.5 yields its $\mathcal{W}^{\mathrm{t}}$-relative compactness. For a cofinal sequence $\mathcal{L}$, let $\hat{\mu}$ be any of the accumulating points of $\{\pi_\Lambda(\cdot|\hat{\xi})\}_{\Lambda\in\mathcal{L}}$. By Lemma 2.13 $\hat{\mu} \in \mathcal{G}^{\mathrm{t}}$ and by Lemma 5.2 $\hat{\mu}$ dominates every element of $\mathrm{ex}(\mathcal{G}^{\mathrm{t}})$. Hence, $\hat{\mu} = \mu_+$ since the maximal element is unique. The same is true for the remaining accumulation points of $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda\in\mathcal{L}}$; thus, for every cofinal sequence $\mathcal{L}$ and for every ℓ_0, we have

$$\lim_{\mathcal{L}} \pi_\Lambda(\cdot| \pm \hat{\xi}) = \mu_\pm. \tag{7.13}$$

Remark 7.5. *As the configuration (7.12) is constant with respect to* $\tau \in S_\beta$, *the kernel* $\pi_\Lambda(\cdot|\hat{\xi})$ *may be considered as the one* $\hat{\pi}_\Lambda(\cdot|0)$ *corresponding to the Hamiltonian with the external field* $\hat{\xi}$, *that is,*

$$H_\Lambda - \sum_{\ell\in\Lambda}(q_\ell, \hat{\xi}_\ell). \tag{7.14}$$

7.3. Reference Models

We shall prove Theorems 3.12, 3.13 by comparing our model with two reference models, defined as follows. Let J and V be the same as in (3.19) and (3.20) respectively. For $\Lambda \Subset \mathbb{L} = \mathbb{Z}^d$, we set (c.f., (2.2))

$$H_\Lambda^{\text{low}} = \sum_{\ell\in\Lambda}\left[H_\ell^{\text{har}} + V(x_\ell)\right] - \frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J\epsilon_{\ell\ell'}x_\ell x_{\ell'}, \quad x_\ell \in \mathbb{R}, \tag{7.15}$$

where H_ℓ^{har} is given by (2.21) and $\epsilon_{\ell\ell'} = 1$ if $|\ell - \ell'| = 1$ and $\epsilon_{\ell\ell'} = 0$ otherwise. The second reference model is defined on an arbitrary $\mathbb{L}$ satisfying (2.1). For $\Lambda \Subset \mathbb{L}$, we set

$$H_\Lambda^{\text{up}} = \sum_{\ell\in\Lambda}\left[H_\ell^{\text{har}} + v\left(x_\ell^2\right)\right] - \frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}x_\ell x_{\ell'} = \sum_{\ell\in\Lambda}\tilde{H}_\ell - \frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}x_\ell x_{\ell'}, \tag{7.16}$$

where $\tilde{H}_\ell$ is defined by (3.31) and the interaction intensities $J_{\ell\ell'}$ are the same as in (2.2). Since both these models are particular cases of the model we consider, their sets of Euclidean Gibbs measures have the properties established by Theorems 3.1 – 3.3. By $\mu_\pm^{\text{low}}$, $\mu_\pm^{\text{up}}$ we denote the corresponding extreme elements.

Remark 7.6. The anharmonic potentials of both reference models have the form (7.3) with the zero external field $h_\ell = 0$ and the functions v_ℓ being convex. Hence, they obey the conditions of all the statements of subsection 7.1. By construction, the *low*-reference model is translation invariant. The *up*-reference model is translation invariant if $\mathbb{L}$ is a lattice and $J_{\ell\ell'}$ are translation invariant.

In the statements below the comparison with the *low*-reference model relates to the case of $\mathbb{L} = \mathbb{Z}^d$.

Lemma 7.7. *For every* ℓ, *it follows that*

$$\mu_+^{\text{low}}(\omega_\ell(0)) \le \mu_+(\omega_\ell(0)) \le \mu_+^{\text{up}}(\omega_\ell(0)). \tag{7.17}$$

Proof: By (7.13) we have that for any $\mathcal{L}$,

$$\int_{\Omega} \omega_\ell(\tau)\mu_{\pm}(\mathrm{d}\omega) = \lim_{\mathcal{L}} \int_{\Omega} \omega_\ell(\tau)\pi_\Lambda(\mathrm{d}\omega| \pm \hat{\xi}), \quad \text{for all } \tau. \tag{7.18}$$

Thus, the proof will be done if we show that for all $\Lambda \Subset \mathbb{L}$ and $\ell \in \Lambda$,

$$\pi_\Lambda^{\mathrm{low}}(\omega_\ell(0)|\hat{\xi}) \leq \pi_\Lambda(\omega_\ell(0)|\hat{\xi}) \leq \pi_\Lambda^{\mathrm{up}}(\omega_\ell(0)|\hat{\xi}). \tag{7.19}$$

First we prove the left-hand inequality in (7.19). For given $\Lambda \Subset \mathbb{L}$ and $t, s \in [0, 1]$, we introduce

$$\begin{aligned} \varpi_\Lambda^{(t,s)}(\mathrm{d}\omega_\Lambda) = \frac{1}{Y(t,s)} \exp \Bigg(&\frac{1}{2} \sum_{\ell,\ell' \in \Lambda} J\epsilon_{\ell\ell'}(\omega_\ell, \omega_{\ell'})_{L^2_\beta} + \sum_{\ell\in\Lambda} \left(\omega_\ell, \eta_\ell^{\ell_0,s}\right)_{L^2_\beta} \\ &- \sum_{\ell\in\Lambda} \int_0^\beta V(\omega_\ell(\tau))\mathrm{d}\tau + \frac{s}{2} \sum_{\ell,\ell'\in\Lambda} [J_{\ell\ell'} - J\epsilon_{\ell\ell'}]\,(\omega_\ell, \omega_{\ell'})_{L^2_\beta} \\ &- t \sum_{\ell\in\Lambda} \int_0^\beta [V_\ell(\omega_\ell(\tau)) - V(\omega_\ell(\tau))]\,\mathrm{d}\tau \Bigg) \chi_\Lambda(\mathrm{d}\omega_\Lambda), \end{aligned} \tag{7.20}$$

where, see (7.12),

$$\begin{aligned} \eta_\ell^{\ell_0,s}(\tau) \stackrel{\mathrm{def}}{=} &\sum_{\ell'\in\Lambda^c} J\epsilon_{\ell\ell'}\hat{\xi}_{\ell'}(\tau) \\ &+ s \sum_{\ell'\in\Lambda^c} [J_{\ell\ell'} - J\epsilon_{\ell\ell'}]\,\hat{\xi}_{\ell'}(\tau) \geq \sum_{\ell'\in\Lambda^c} J\epsilon_{\ell\ell'}\hat{\xi}_{\ell'}(\tau) > 0, \end{aligned} \tag{7.21}$$

which in fact is independent of τ, and

$$\begin{aligned} Y(t,s) = \int_{\Omega_\Lambda} \exp \Bigg(&\frac{1}{2} \sum_{\ell,\ell' \in \Lambda} J\epsilon_{\ell\ell'}(\omega_\ell, \omega_{\ell'})_{L^2_\beta} + \sum_{\ell\in\Lambda} \left(\omega_\ell, \eta_\ell^{\ell_0,s}\right)_{L^2_\beta} \\ &- \sum_{\ell\in\Lambda} \int_0^\beta V(\omega_\ell(\tau))\mathrm{d}\tau + \frac{s}{2} \sum_{\ell,\ell'\in\Lambda} [J_{\ell\ell'} - J\epsilon_{\ell\ell'}]\,(\omega_\ell, \omega_{\ell'})_{L^2_\beta} \\ &- t \sum_{\ell\in\Lambda} \int_0^\beta [V_\ell(\omega_\ell(\tau)) - V(\omega_\ell(\tau))]\,\mathrm{d}\tau \Bigg) \chi_\Lambda(\mathrm{d}\omega_\Lambda). \end{aligned}$$

Since the site-dependent 'external field' (7.21) is positive, the moments of the measure (7.20) obey the GKS inequalities. Therefore, for any $\ell \in \Lambda$, the function

$$\phi(t,s) = \varpi_\Lambda^{(t,s)}(\omega_\ell(0)), \quad t, s \in [0, 1], \tag{7.22}$$

is continuous and increasing in both variables. Indeed, taking into account (3.19), (3.20), and (3.23), we get

$$
\begin{aligned}
\frac{\partial}{\partial s}\phi(t,s) = & \sum_{\ell'\in\Lambda}\left[J_{\ell\ell'} - J\epsilon_{\ell\ell'}\right]\hat{\xi}_{\ell'}(0) \\
& \times \int_0^\beta \left\{\varpi_\Lambda^{(t,s)}\left[\omega_\ell(0)\omega_{\ell'}(\tau)\right] - \varpi_\Lambda^{(t,s)}\left[\omega_\ell(0)\right]\cdot\varpi_\Lambda^{(t,s)}\left[\omega_{\ell'}(\tau)\right]\right\}\mathrm{d}\tau \\
& + \frac{1}{2}\sum_{\ell_1,\ell_2\in\Lambda}\left[J_{\ell_1\ell_2} - J\epsilon_{\ell_1\ell_2}\right]\left\{\varpi_\Lambda^{(t,s)}\left[\omega_\ell(0)(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta}\right]\right. \\
& \left. - \varpi_\Lambda^{(t,s)}\left[\omega_\ell(0)\right]\cdot\varpi_\Lambda^{(t,s)}\left[(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta}\right]\right\} \geq 0, \\
\frac{\partial}{\partial t}\phi(t,s) = & \sum_{\ell'\in\Lambda}\int_0^\beta \left\{\varpi_\Lambda^{(t,s)}\left(\omega_\ell(0)\cdot\left[V(\omega_{\ell'}(\tau)) - V_{\ell'}(\omega_{\ell'}(\tau))\right]\right)\right. \\
& \left. - \varpi_\Lambda^{(t,s)}\left[\omega_\ell(0)\right]\cdot\varpi_\Lambda^{(t,s)}\left[V(\omega_{\ell'}(\tau)) - V_{\ell'}(\omega_{\ell'}(\tau))\right]\right\}\mathrm{d}\tau \geq 0.
\end{aligned}
$$

But by (7.20) and (7.22)

$$\phi(0,0) = \pi_\Lambda^{\mathrm{low}}(\omega_\ell(0)), \quad \phi(1,1) = \pi_\Lambda(\omega_\ell(0)),$$

which proves the left-hand inequality in (7.19). To prove the right-hand one we have to take the measure (7.20) with $s = 1$ and $v(x_\ell^2)$ instead of $V(x_\ell)$ and repeat the above steps taking into account (3.30). □

In the next statement we summarize the properties of the reference models.

Corollary 7.8. (Comparison Criterion) *The model considered undergoes a phase transition if the* low-*reference model does so. The uniqueness of tempered Euclidean Gibbs measures of the* up-*reference model implies that* $|\mathcal{G}^{\mathrm{t}}| = 1$.

Proof: The proof follows immediately from (7.17) and Lemma 6.3. □

7.4. Estimates for Pair Correlation Functions

For $\Delta \subset \Lambda$, $\ell, \ell' \in \Lambda$, $\tau, \tau' \in [0, \beta]$, and $t \in [0, 1]$, we set

$$Q_{\ell\ell'}^{\Lambda}(\tau,\tau'|\Delta,t) = \int_{\Omega_\Lambda}\omega_\ell(\tau)\omega_{\ell'}(\tau')\varpi_{\Lambda,\Delta}^{(t)}(\mathrm{d}\omega_\Lambda), \tag{7.23}$$

where this time we have denoted

$$\varpi^{(t)}_{\Lambda,\Delta}(\mathrm{d}\omega_\Lambda) = \frac{1}{Y_{\Lambda,\Delta}(t)} \exp\left\{ \frac{1}{2} \sum_{\ell_1,\ell_2 \in \Lambda\setminus\Delta} J_{\ell_1\ell_2}(\omega_{\ell_1}, \omega_{\ell_2})_{L^2_\beta} \right.$$
$$+ t \left(\sum_{\ell_1\in\Delta} \sum_{\ell_2 \in \Lambda\setminus\Delta} J_{\ell_1\ell_2}(\omega_{\ell_1}, \omega_{\ell_2})_{L^2_\beta} + \frac{1}{2} \sum_{\ell_1,\ell_2\in\Delta} J_{\ell_1\ell_2}(\omega_{\ell_1}, \omega_{\ell_2})_{L^2_\beta} \right)$$
$$\left. - \sum_{\ell\in\Lambda} \int_0^\beta V_\ell(\omega_\ell(\tau))\, \mathrm{d}\tau \right\} \chi_\Lambda(\mathrm{d}\omega_\Lambda),$$

$$Y_{\Lambda,\Delta}(t) = \int_{\Omega_\Lambda} \exp\left\{ \frac{1}{2} \sum_{\ell_1,\ell_2 \in \Lambda\setminus\Delta} J_{\ell_1\ell_2}(\omega_{\ell_1}, \omega_{\ell_2})_{L^2_\beta} \right.$$
$$+ t \left(\sum_{\ell_1\in\Delta} \sum_{\ell_2 \in \Lambda\setminus\Delta} J_{\ell_1\ell_2}(\omega_{\ell_1}, \omega_{\ell_2})_{L^2_\beta} + \frac{1}{2} \sum_{\ell_1,\ell_2\in\Delta} J_{\ell_1\ell_2}(\omega_{\ell_1}, \omega_{\ell_2})_{L^2_\beta} \right)$$
$$\left. - \sum_{\ell\in\Lambda} \int_0^\beta V_\ell(\omega_\ell(\tau))\, \mathrm{d}\tau \right\} \chi_\Lambda(\mathrm{d}\omega_\Lambda). \tag{7.24}$$

By literal repetition of the arguments used for proving Lemma 7.7 one proves the following

Proposition 7.9. *The above $Q^\Lambda_{\ell\ell'}(\tau, \tau'|\Delta, t)$ is an increasing continuous function of $t \in [0, 1]$.*

Corollary 7.10. *Let the conditions of Proposition 7.2 be satisfied. Then for any pair $\Lambda \subset \Lambda' \Subset \mathbb{L}$ and for all τ and ℓ, the functions (7.2) obey the estimate*

$$K^\Lambda_{\ell\ell'}(\tau, \tau'|0) \leq K^{\Lambda'}_{\ell\ell'}(\tau, \tau'|0), \tag{7.25}$$

which holds for all $\ell, \ell' \in \Lambda$ and $\tau, \tau' \in [0, \beta]$.

Now we obtain bounds for the correlation functions of the reference models for a one-point $\Lambda = \{\ell\}$. Set

$$K^{\mathrm{up}}_\ell(\tau, \tau') = \pi^{\mathrm{up}}_\ell(\omega_\ell(\tau)\omega_\ell(\tau')|0), \quad K^{\mathrm{low}}_\ell(\tau, \tau') = \pi^{\mathrm{low}}_\ell(\omega_\ell(\tau)\omega_\ell(\tau')|0), \tag{7.26}$$

We recall that the parameter Δ was defined by (3.32).

Lemma 7.11. *For every β, it follows that*

$$K_\ell^{\rm up} \stackrel{\rm def}{=} \int_0^\beta K_\ell^{\rm up}(\tau, \tau')\,{\rm d}\tau \leq 1/m\Delta^2. \tag{7.27}$$

Proof: In view of (2.14) the above integral is independent of τ. By (2.13) and (2.15)

$$K_\ell^{\rm up} = \frac{1}{\tilde{Z}_\ell}\int_0^\beta {\rm trace}\big\{x_\ell e^{-\tau \tilde{H}_\ell} x_\ell e^{-(\beta-\tau)\tilde{H}_\ell}\big\}{\rm d}\tau, \quad \tilde{Z}_\ell = {\rm trace}[e^{-\beta\tilde{H}_\ell}], \tag{7.28}$$

where the Hamiltonian $\tilde{H}$ was defined in (3.31). Its spectrum $\{E_n\}_{n\in\mathbb{N}}$ determines by (3.32) the parameter Δ. Integrating in (7.28) we get

$$\begin{aligned} K_\ell^{\rm up} &= \frac{1}{\tilde{Z}_\ell}\sum_{n,n'\in\mathbb{N}_0,\ n\neq n'} \big|(\psi_n, x_\ell\psi_{n'})_{L^2(\mathbb{R})}\big|^2 \frac{(E_n - E_{n'})(e^{-\beta E_{n'}} - e^{-\beta E_n})}{(E_n - E_{n'})^2} \\ &\leq \frac{1}{\tilde{Z}_\ell}\cdot\frac{1}{\Delta^2}\sum_{n,n'\in\mathbb{N}_0} \big|(\psi_n, x_\ell\psi_{n'})_{L^2(\mathbb{R})}\big|^2 (E_n - E_{n'})(e^{-\beta E_{n'}} - e^{-\beta E_n}) \\ &= \frac{1}{\Delta^2}\cdot\frac{1}{\tilde{Z}_\ell}{\rm trace}\{[x_\ell, [\tilde{H}_\ell, x_\ell]]e^{-\beta\tilde{H}_\ell}\} = \frac{1}{m\Delta^2}, \end{aligned} \tag{7.29}$$

where ψ_n, $n \in \mathbb{N}_0$ are the eigenfunctions of $\tilde{H}_\ell$ and $[\cdot,\cdot]$ stands for commutator. □

For the functions $K_\ell^{\rm low}$, a representation like (7.28) is obtained by means of the following Hamiltonian

$$\hat{H}_\ell = H_\ell^{\rm har} + V(x_\ell) = -\frac{1}{2m}\left(\frac{\partial}{\partial x_\ell}\right)^2 + \frac{a}{2}x_\ell^2 + V(x_\ell), \tag{7.30}$$

where m and a are the same as in (3.31) but V is given by (3.20). Thereby,

$$K_\ell^{\rm low}(0,0) = {\rm trace}\big[x_\ell^2\exp(-\beta\hat{H}_\ell)\big]/{\rm trace}[\exp(-\beta\hat{H}_\ell)] \stackrel{\rm def}{=} \hat{\varrho}(x_\ell^2). \tag{7.31}$$

Lemma 7.12. *Let t_* be the solution of (3.22). Then $K_\ell^{\rm low}(0,0) \geq t_*$.*

Proof: By Bogoliubov's inequality (see e.g., Ref. 81), it follows that

$$\hat{\varrho}_\ell([p_\ell, [\hat{H}_\ell, p_\ell]]) \geq 0, \quad p_\ell = -\sqrt{-1}\frac{\partial}{\partial x_\ell},$$

which by (3.20), (3.21) yields

$$
\begin{aligned}
& a + 2b^{(1)} + \sum_{s=2}^{r} 2s(2s-1)b^{(s)}\hat{\varrho}\left[x_\ell^{2(s-1)}\right] \\
& \quad = a + 2b^{(1)} + \sum_{s=2}^{r} 2s(2s-1)b^{(s)}\pi_\ell^{\rm low}\left[(\omega_\ell(0))^{2(s-1)}\right] \geq 0.
\end{aligned}
$$

Now we use the Gaussian domination inequality (7.10) and obtain $K_\ell^{\rm low} \geq t_*$. □

7.5. Periodic States and Proof of Theorem 3.12

In view of Corollary 7.8 to prove Theorem 3.12 we show that

$$
\mu_+^{\rm low}(\omega_\ell(0)) > 0, \tag{7.32}
$$

if the conditions of Theorem 3.12 are satisfied. To this end we employ the translation invariance and reflection positivity of the low-reference model. With this connection we construct periodic Euclidean Gibbs states by introducing (c.f., (2.30))

$$
I_\Lambda^{\rm per}(\omega_\Lambda) = -\frac{J}{2}\sum_{\ell,\ell'\in\Lambda} \epsilon_{\ell\ell'}^{\Lambda}(\omega_\ell, \omega_{\ell'})_{L_\beta^2} + \sum_{\ell\in\Lambda}\int_0^\beta V(\omega_\ell(\tau))\,{\rm d}\tau, \tag{7.33}
$$

where

$$
\Lambda = (-L, L]^d \bigcap \mathbb{L}, \quad L \in \mathbb{N}, \tag{7.34}
$$

and $\epsilon_{\ell\ell'}^{\Lambda} = 1$ if $|\ell - \ell'|_\Lambda = 1$ and $\epsilon_{\ell\ell'}^{\Lambda} = 0$ otherwise. Here

$$
\begin{aligned}
|\ell - \ell'|_\Lambda &= [|\ell_1 - \ell_1'|_L^2 + \cdots + |\ell_d - \ell_d'|_L^2]^{1/2}, \\
|\ell_j - \ell_j'|_L &= \min\{|\ell_j - \ell_j'|; L - |\ell_j - \ell_j'|\}, \qquad j = 1, \ldots, d.
\end{aligned}
$$

Clearly, $I_\Lambda^{\rm per}$ is invariant with respect to the translations of the torus which one obtains by identifying the opposite walls of the box (7.34). The energy functional $I_\Lambda^{\rm per}$ corresponds to the following periodic Hamiltonian

$$
H_\Lambda^{\rm per} = \sum_{\ell\in\Lambda}\left[H_\ell^{\rm har} + V(x_\ell)\right] - \frac{J}{2}\sum_{\ell,\ell'\in\Lambda}\epsilon_{\ell\ell'}^{\Lambda} x_\ell x_{\ell'}, \tag{7.35}
$$

in the same sense as I_Λ given by (2.30) corresponds to H_Λ given by (2.2). Now we introduce the periodic kernels (c.f., (2.54))

$$
\pi_\Lambda^{\rm per}({\rm d}\omega) = \frac{1}{Z_\Lambda^{\rm per}}\exp\left[-I_\Lambda^{\rm per}(\omega_\Lambda)\right]\chi_\Lambda({\rm d}\omega_\Lambda)\prod_{\ell\in\Lambda^c}\delta({\rm d}\omega_\ell), \tag{7.36}
$$

where δ is the Dirac measure concentrated at $\omega_\ell = 0$ and

$$Z_\Lambda^{\rm per} = \int_{\Omega_\Lambda} \exp\left[-I_\Lambda^{\rm per}(\omega_\Lambda)\right] \chi_\Lambda(d\omega_\Lambda).$$

Thereby, for every box Λ, the above $\pi_\Lambda^{\rm per}$ is a probability measure on $\Omega^{\rm t}$. By $\mathcal{L}_{\rm box}$ we denote the sequence of boxes (7.34) indexed by $L \in \mathbb{N}$. For a given $\alpha \in \mathcal{I}$, let us choose $\vartheta, \varkappa > 0$ such that the estimate (4.13) holds.

Lemma 7.13. *For every box Λ, $\alpha \in \mathcal{I}$, and $\sigma \in (0, 1/2)$, the measure $\pi_\Lambda^{\rm per}$ obeys the estimate*

$$\int_\Omega \|\omega\|^2_{\alpha,\sigma} \pi_\Lambda^{\rm per}(d\omega) \leq C_{7.37}. \tag{7.37}$$

Thereby, the sequence $\{\pi_\Lambda^{\rm per}\}_{\Lambda\in\mathcal{L}_{\rm box}}$ is $\mathcal{W}^{\rm t}$-relatively compact.

Proof: For $\ell \in \Lambda$ such that $\{\ell' \in \mathbb{L} \mid |\ell - \ell'| = 1\} \subset \Lambda$, we set $\Delta_\ell = \mathbb{L} \setminus \{\ell\}$. Then let ν_ℓ^Λ be the projection of $\pi_\Lambda^{\rm per}$ onto $\mathcal{B}(\Omega_{\Delta_\ell})$. Let also $\nu_\ell(\cdot|\xi)$, $\xi \in \Omega$ be the following probability measure on the single-spin space $\Omega_{\{\ell\}} = C_\beta$

$$\nu_\ell(d\omega_\ell|\xi) = \frac{1}{N_\ell(\xi)} \exp\left\{J\sum_{\ell'} \epsilon_{\ell\ell'}(\omega_\ell, \xi_{\ell'})_{L^2_\beta} - \int_0^\beta V(\omega_\ell(\tau))\, d\tau\right\} \chi(d\omega_\ell). \tag{7.38}$$

Then (c.f., (2.56)) desintegrating $\pi_\Lambda^{\rm per}$ we get

$$\pi_\Lambda^{\rm per}(d\omega) = \nu_\ell(d\omega_\ell|\omega_{\Delta_\ell})\nu_\ell^\Lambda(d\omega_{\Delta_\ell}). \tag{7.39}$$

Like in Lemma 4.1 and Corollary 4.2 one proves that the measure $\nu_\ell(\cdot|\xi)$ obeys

$$\int_{C_\beta} \exp\left\{\lambda_\sigma|\omega_\ell|^2_{C^\sigma_\beta} + \varkappa|\omega_\ell|^2_{L^2_\beta}\right\}\nu_\ell(d\omega_\ell|\omega_{\Delta_\ell}) \leq \exp\left\{C_{4.1} + \vartheta J \sum_{\ell'} \epsilon_{\ell\ell'}|\omega_{\ell'}|^2_{L^2_\beta}\right\},$$

where λ_σ, $\varkappa$, and ϑ are as in (4.1), (4.4). Now we integrate both sides of this inequality with respect to ν_ℓ^Λ and get, c.f., (4.12), (4.13)

$$n_\ell^{\rm per}(\Lambda) \stackrel{\rm def}{=} \log\left\{\int_\Omega \exp\left[\lambda_\sigma|\omega_\ell|^2_{C^\sigma_\beta} + \varkappa|\omega_\ell|^2_{L^2_\beta}\right]\pi_\Lambda^{\rm per}(d\omega)\right\} \leq C_{4.7}.$$

Then the estimate (7.37) is obtained in the same way as (4.16) was proven. The relative $\mathcal{W}_\alpha$-compactness of $\{\pi_\Lambda^{\rm per}\}_{\Lambda\in\mathcal{L}_{\rm per}}$ follows from (7.37) and the compactness of the embeddings $\Omega_{\alpha,\sigma} \hookrightarrow \Omega_{\alpha'}$, $\alpha < \alpha'$. The $\mathcal{W}^{\rm t}$-compactness is a consequence of Lemma 4.5. □

Lemma 7.14. *Every $\mathcal{W}^{\rm t}$-accumulation point $\mu^{\rm per}$ of the sequence $\{\pi_\Lambda^{\rm per}\}_{\Lambda\in\mathcal{L}_{\rm per}}$ is a Euclidean Gibbs measure of the* low-*reference model.*

Proof: Let $\mathcal{L} \subset \mathcal{L}_{\rm per}$ be the subsequence along which $\{\pi_\Lambda^{\rm per}\}_{\Lambda\in\mathcal{L}}$ converges to $\mu^{\rm per} \in \mathcal{P}(\Omega^{\rm t})$. Then $\{\nu_\ell^\Lambda\}_{\Lambda\in\mathcal{L}}$ converges to the projection of $\mu^{\rm per}$ on $\mathcal{B}(\Omega_{\Delta_\ell})$. Employing the Feller property (Lemma 2.10) we pass in (7.39) to the limit along this $\mathcal{L}$ and apply both its sides to a function $f \in C_{\rm b}(\Omega^{\rm t})$. This yields that $\mu^{\rm per}$ has the same one-point conditional distributions as the Euclidean Gibbs measures of the reference model. But according to Theorem 1.33 of Ref. 36, page 23, every Gibbs measure is uniquely defined by its conditional distributions corresponding to one-point sets $\Lambda = \{\ell\}$ only. □

Now we are at a position to prove that (7.32) holds if $\beta > \beta_*$. Given a box Λ, we introduce

$$P_\Lambda(\beta) = \int_\Omega \left| \frac{1}{\beta|\Lambda|} \sum_{\ell\in\Lambda} \int_0^\beta \omega_\ell(\tau)\,{\rm d}\tau \right|^2 \pi_\Lambda^{\rm per}({\rm d}\omega). \tag{7.40}$$

For any ℓ, one can take the box Λ such that the Euclidean distance from this ℓ to Λ^c be greater than 1. Then by Corollary 7.10 and Lemma 7.12 one gets

$$\int_\Omega [\omega_\ell(0)]^2 \pi_\Lambda^{\rm per}({\rm d}\omega) \geq K_\ell^{\rm low}(0,0) \geq t_*. \tag{7.41}$$

The infrared estimates based on the reflection positivity of the *low*-reference model, together with the Bruch-Falk inequality[6] and the estimate (7.41), lead to the following bound

$$P_\Lambda(\beta) \geq t_* f(\beta/4mt_*) - \theta_d/2\beta J, \tag{7.42}$$

which holds for any box Λ. By means of the Griffiths theorem, see Ref. 29, Theorem 1.1 and the corollaries, one can prove that

$$\mu^{\rm per}(\omega_\ell(0)) \geq \limsup_{\mathcal{L}_{\rm per}} \sqrt{P_\Lambda(\beta)}. \tag{7.43}$$

Therefore, the estimate (7.32) holds if the right-hand side of (7.43) is positive, which can be ensured by taking $\beta > \beta_*$, see (3.26) and (3.27), (3.28).

7.6. Proof of Theorem 3.13

Now we make precise the parameter δ participating in the condition (2.40). In what follows, we set $\delta = m\Delta^2$, where the parameter Δ was defined by (3.32). Then

$$\hat{J}_0 < \hat{J}_\alpha < m\Delta^2. \tag{7.44}$$

[6] See Theorem VI.7.5, page 392 of Ref. 81 or Theorem 3.1 in Ref. 29

Let us consider the examples following Assumption 2.5. If $J_{\ell\ell'}$ obeys (2.41), the values of α in question exist in view of

$$\lim_{\alpha\to 0+} \hat{J}_\alpha = \hat{J}_0, \tag{7.45}$$

which readily follows from (2.41), (2.42). If the weights are chosen as in (2.44), one can use ε to ensure (7.44). Indeed, simple calculations yield

$$0 < \hat{J}^{(\varepsilon)}_\alpha - \hat{J}_0 \leq \varepsilon\alpha d \hat{J}^{(1)}_\alpha,$$

where to indicate the dependence of $\hat{J}_\alpha$ on ε we write $\hat{J}^{(\varepsilon)}_\alpha$. Thereby, we fix $\alpha \in \mathcal{I}$ and choose ε to obey $\varepsilon < m\Delta^2/\alpha d\hat{J}^{(1)}_\alpha$.

Now let us turn to the proof of Theorem 3.13. By Corollary 7.8 it is enough to prove the uniqueness for the *up*-reference model, which by Lemma 6.3 is equivalent to

$$\mu^{\rm up}_{+}(\omega_\ell(0)) = 0, \quad \text{for all} \quad \beta > 0 \ \text{ and } \ \ell. \tag{7.46}$$

Given $\Lambda \Subset \mathbb{L}$, we introduce the matrix $(T^\Lambda_{\ell\ell'})_{\ell,\ell'\in\mathbb{L}}$ as follows. We set $T^\Lambda_{\ell\ell'} = 0$ if either of ℓ, ℓ' belongs to Λ^c. For $\ell, \ell' \in \Lambda$,

$$T^\Lambda_{\ell\ell'} = \sum_{\ell_1\in\Lambda} J_{\ell\ell_1} \int_0^\beta \pi^{\rm up}_\Lambda \left[\omega_{\ell_1}(\tau)\omega_{\ell'}(\tau')\,|0\right] {\rm d}\tau'. \tag{7.47}$$

By (2.14) the above integral is independent of τ.

Lemma 7.15. *If (3.33) is satisfied, there exists $\alpha \in \mathcal{I}$, such that for every $\Lambda \Subset \mathbb{L}$, the matrix $(T^\Lambda_{\ell\ell'})_{\ell,\ell'\in\mathbb{L}}$ defines a bounded operator in the Banach space $l^\infty(w_\alpha)$.*

Proof: The proof will be based on a generalization of the method used in Ref. 5 for proving Lemma 4.7. For $t \in [0, 1]$, let $\varpi^{(t)}_\Lambda \in \mathcal{P}(\Omega_\Lambda))$ be defined by (7.24) with $\Delta = \Lambda$ and each $V_\ell(\omega_\ell(\tau))$ replaced by $v([\omega_\ell(\tau)]^2)$, where v is the same as in (3.31). Then by (7.16)

$$\varpi^{(0)}_\Lambda = \prod_{\ell\in\Lambda} \pi^{\rm up}_\ell(\cdot|0), \quad \varpi^{(1)}_\Lambda = \pi^{\rm up}_\Lambda(\cdot|0), \quad \text{for any } \ \Lambda \Subset \mathbb{L}. \tag{7.48}$$

Thereby, we set

$$T^\Lambda_{\ell\ell'}(t) = \sum_{\ell_1} J_{\ell\ell_1} \int_0^\beta \varpi^{(t)}_\Lambda \left[\omega_{\ell_1}(\tau)\omega_{\ell'}(\tau')\right] {\rm d}\tau' \quad t \in [0, 1]. \tag{7.49}$$

One can show that for every fixed ℓ, ℓ', the above $T^\Lambda_{\ell\ell'}(t)$ is differentiable on the interval $t \in (0, 1)$ and continuous at its endpoints, where (see (7.27))

$$T^\Lambda_{\ell\ell'}(0) = J_{\ell\ell'} K^{\rm up}_{\ell'} \leq J_{\ell\ell'}/m\Delta^2, \qquad T^\Lambda_{\ell\ell'}(1) = T^\Lambda_{\ell\ell'}. \tag{7.50}$$

Computing the derivative we get

$$\frac{\partial}{\partial t} T^{\Lambda}_{\ell\ell'}(t) = \frac{1}{2} \sum_{\ell_1,\ell_2,\ell_3} J_{\ell\ell_1} J_{\ell_2\ell_3} \int_0^{\beta} \int_0^{\beta} U^{\Lambda}_{\ell\ell'\ell_2\ell_3}(t, \tau, \tau', \tau_1, \tau_1)\, \mathrm{d}\tau' \mathrm{d}\tau_1 \qquad (7.51)$$
$$+ \sum_{\ell_1} T^{\Lambda}_{\ell\ell_1}(t) T^{\Lambda}_{\ell_1\ell'}(t),$$

where $U^{\Lambda}_{\ell\ell'\ell_1\ell_2}(t, \tau, \tau', \tau_1, \tau_1)$ is the Ursell function which obeys the estimate (7.11) since the function v is convex. Hence, except for the trivial case $J_{\ell\ell'} \equiv 0$, the first term in (7.51) is strictly negative. Let us consider the following Cauchy problem

$$\frac{\partial}{\partial t} L_{\ell\ell'}(t) = \sum_{\ell_1} L_{\ell\ell_1}(t) L_{\ell_1\ell'}(t), \quad L_{\ell\ell'}(0) = \lambda J_{\ell\ell'}, \quad \ell, \ell' \in \mathbb{L}, \qquad (7.52)$$

where $\lambda \in (1/m\Delta^2, 1/\hat{J}_\alpha)$, with $\alpha \in \mathcal{I}$ chosen to obey (7.44). For such α, one can solve the problem (7.52) in the space $l^\infty(w_\alpha)$ (see Remark 2.6) and obtain

$$L(t) = \lambda J \left[I - \lambda t J \right]^{-1}, \quad \|L(t)\|_{l^\infty(w_\alpha)} \leq \frac{\lambda \hat{J}_\alpha}{1 - \lambda t \hat{J}_\alpha}. \qquad (7.53)$$

where I is the identity operator. Now let us compare (7.51) and (7.52) considering the former expression as a differential equation subject to the initial condition (7.50). Since the first term in (7.51) is negative, one can apply Theorem V, page 65 of Ref. 88 and obtain $T^{\Lambda}_{\ell\ell'} < L_{\ell\ell'}(1)$, which in view of (7.53) yields the proof. □

Proof of Theorem 3.13: For $\ell, \ell_0, \Lambda \Subset \mathbb{L}$, such that $\ell \in \Lambda$, and $t \in [0, 1]$, we set

$$\psi_\Lambda(t) = \int_\Omega \omega_\ell(0) \pi^{\mathrm{up}}_\Lambda(\mathrm{d}\omega | t\xi^{\ell_0}), \qquad (7.54)$$

where ξ^{ℓ_0} is the same as in (7.12). The function ψ_Λ is obviously differentiable on the interval $t \in (-1, 1)$ and continuous at its endpoints. Then

$$0 \leq \psi_\Lambda(1) \leq \sup_{t \in [0,1]} \psi'_\Lambda(t). \qquad (7.55)$$

The derivative is

$$\psi'_\Lambda(t) = \sum_{\ell_1 \in \Lambda,\ \ell_2 \in \Lambda^c} J_{\ell\ell_1} \int_0^\beta \pi^{\mathrm{up}}_\Lambda \left[\omega_{\ell_1}(0) \omega_{\ell_2}(\tau) \left| t\xi^{\ell_0} \right. \right] \eta_{\ell_2} \mathrm{d}\tau, \qquad (7.56)$$

where the 'external field' $\eta_{\ell'} = [b \log(1 + |\ell' - \ell_0|)]^{1/2}$ is positive at each site. Thus, we may use (7.8) and obtain

$$\psi'_\Lambda(t) \leq \sum_{l' \in \Lambda^c} T^{\Lambda}_{\ell\ell'} \eta_{\ell'}. \qquad (7.57)$$

By Assumption 2.5 (b), $\eta \in l^\infty(w_\alpha)$ with any $\alpha > 0$, then employing Lemma 7.15, the estimate (7.53) in particular, we conclude that the right-hand side of (7.57) tends to zero as $\Lambda \nearrow \mathbb{L}$, which by (7.18) and (7.54), (7.55) yields (7.46). □

8. UNIQUENESS AT NONZERO EXTERNAL FIELD

In statistical mechanics phase transitions may be associated with nonanalyticity of thermodynamic characteristics considered as functions of the external field h. In special cases one can oversee at which values of h this nonanaliticity can occur. The Lee-Yang theorem states that the only such value is $h = 0$; hence, no phase transitions can occur at nonzero h. In the theory of classical lattice models these arguments were applied in Refs. 60–62. We refer also to sections 4.5, 4.6 in Ref. 37 and sections IX.3–IX.5 in Ref. 79 where applications of such arguments in quantum field theory and classical statistical mechanics are discussed.

In the case of lattice models with the single-spin space $\mathbb{R}$ the validity of the Lee-Yang theorem depends on the properties of the anharmonic potentials. For the polynomials $V(x) = x^4 + ax^2$, $a \in \mathbb{R}$, the Lee-Yang theorem holds, see e.g., Theorem IX.15 on page 342 in Ref. 79. But no other examples of this kind were known, see the discussion on page 71 in Ref. 37. Below we give a sufficient condition for the potentials V to have the corresponding property and discuss some examples. Here we use the family $\mathcal{F}_{\rm Laguerre}$ defined by (3.35). We also prove a number of lemmas, which allow us to apply the arguments based on the Lee-Yang theorem to our quantum model and hence to prove Theorem 3.14.

Recall that the elements of $\mathcal{F}_{\rm Laguerre}$ can be continued to entire functions $\varphi : \mathbb{C} \to \mathbb{C}$, which have no zeros outside of $(-\infty, 0]$.

Definition 8.1. A probability measure ν on the real line is said to have the Lee-Yang property if there exists $\varphi \in \mathcal{F}_{\rm Laguerre}$ such that

$$\int_{\mathbb{R}} \exp(xy)\nu(\mathrm{d}y) = \varphi(x^2).$$

In Ref. 52, see also Theorem 2.3 in Ref. 56, the following fact was proven.

Proposition 8.2. *Let the function $u : \mathbb{R} \to \mathbb{R}$ be such that for a certain $b \geq 0$, its derivative obeys the condition $b + u' \in \mathcal{F}_{\rm Laguerre}$. Then the probability measure*

$$\nu(\mathrm{d}y) = C \exp[-u(y^2)]\,\mathrm{d}y, \tag{8.1}$$

has the Lee-Yang property.

This statement gives a sufficient condition, the lack of which was mentioned on page 71 of Ref. 37. The example of a polynomial given there for which the corresponding classical models undergo phase transitions at nonzero h, in our notations is $u(t) = t^3 - 2t^2 + (\alpha + 1)t$, $\alpha > 0$. It certainly does not meet the condition of Proposition 8.2. Turning to the model described by Theorem 3.14 we note that, for $v(t) = t^3 + b^{(2)}t^2 + b^{(1)}t$, the function $u(t) = v(t) + at/2$ obeys the conditions of Proposition 8.2 if and only if $b^{(2)} \geq 0$ and $b^{(1)} + a/2 \leq [b^{(2)}]^2/3$. Therefore, according to Theorem 3.14 we have $|\mathcal{G}^{\mathrm{t}}| = 1$ at $h \neq 0$ and $2b^{(1)} + a < 0$, $b^{(2)} \geq 0$. On the other hand, for this model, by Theorem 3.12 one has a phase transition at $h = 0$ and the same coefficients of v.

Set

$$f(h^2) = \int_{\mathbb{R}^n} \exp\left[h \sum_{i=1}^{n} x_i + \sum_{i,j=1}^{n} M_{ij} x_i x_j \right] \prod_{i=1}^{n} \nu(\mathrm{d}x_i), \quad h \in \mathbb{R}. \tag{8.2}$$

By Theorem 3.2 of Ref. 63, we have the following

Proposition 8.3. *If in (8.2) $M_{ij} \geq 0$ for all $i, j = 1, \ldots, n$, and the measure ν is as in Proposition 8.2, then the function f, if exists, belongs to $\mathcal{F}_{\mathrm{Laguerre}}$. It certainly exists if u' is not constant.*

Now let the potential V obey the conditions of Theorem 3.14. Recall that $p_\Lambda(h)$ stands for the pressure (3.13) with $\xi = 0$. Define

$$\varphi_\Lambda(h^2) = p_\Lambda(h), \quad h \in \mathbb{R}. \tag{8.3}$$

Lemma 8.4. *If V obeys the conditions of Theorem 3.14, the function* $\exp(|\Lambda|\varphi_\Lambda)$ *belongs to $\mathcal{F}_{\mathrm{Laguerre}}$.*

Proof: With the help of the lattice approximation technique the function $\exp(|\Lambda|\varphi_\Lambda)$ may be approximated by f_N, $N \in \mathbb{N}$, having the form (8.2) with the measures ν having the form (8.1) with $u(t) = v(t) + at/2$, v is as in (3.36), and non-negative M_{ij} (see Theorem 5.2 in Ref. 4). For every $h \in \mathbb{R}$, $f_N(h^2) \to \exp\left(|\Lambda|\varphi_\Lambda(h^2)\right)$ as $N \to +\infty$. The entire functions f_N are ridge, with the ridge being $[0, +\infty)$. For sequences of such functions, their point-wise convergence on the ridge implies via the Vitali theorem (see e.g., Ref. 79) the uniform convergence on compact subsets of $\mathbb{C}$, which yields the property stated (for more details, see Refs. 53, 57). $\square$

Proof of Theorem 3.14: By Lemma 8.4, for every $\Lambda \Subset \mathbb{L}$, $p_\Lambda(h)$ can be extended to a function of $h \in \mathbb{C}$, holomorphic in the right and left open half-planes. By standard arguments, see e.g., Lemma 39, page 34 of Ref. 53, and Lemma 6.4 it follows that the limit of such extensions $p(h)$ is holomorphic in certain subsets of those half-planes containing the real line, except possibly for the point $h = 0$.

Therefore, $p(h)$ is differentiable at each $h \neq 0$. Then the proof of the theorem follows from Corollary 3.11. $\square$

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